The Binomial Recurrence

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The solution to the recurrence

$$\left|{n \atop k}\right| = \left|{n-1 \atop k}\right| + \left|{n-1 \atop k-1}\right| \tag{1}$$

with boundary conditions $\left|{0 \atop 0}\right| = 1$ and $\left|{0 \atop k}\right| = 0$ for $k \neq 0$ is of course the binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Such a well-known fact can (not surprisingly) be proved in a variety of ways. One is by substitution of the factorial expressions into the recurrence and verifying that the latter is satisfied. A second uses the combinatorial interpretation of the binomial coefficients (see, for example, Benjamin and Quinn [**1**, p. 64]). These are both standard exercises in introductory combinatorics texts. A somewhat more sophisticated method applies generating functions to derive the binomial theorem, at which point Taylor's formula can be used to show that the solution to the recurrence is $\left|{n \atop k}\right| = \frac{n!}{k!(n-k)!}$ (see Wilf [**7**, p. 14]).

There is something a bit unsatisfactory about these techniques, however. The first two methods essentially require one to know (or at least conjecture) the solution to the recurrence in advance. These approaches amount to verifying that the solution is what we already know it is. The third does derive the answer directly from the recurrence, but it invokes the machinery of generating functions and Taylor's formula in order to do so. It seems there should be a method for finding the solution to Equation (1) that does not require that we already know the answer and that only uses basic properties of recurrence relations.

The purpose of this note is to provide such a solution. Specifically, we give an apparently new direct derivation of the solution $\left|{n \atop k}\right| = \frac{n!}{k!(n-k)!}$ to Equation (1), using only basic properties of two-variable triangular recurrence relations.

First, some notation. We use the Iverson bracket $[P]$, which evaluates to 1 if statement P is true and 0 if P is false. We consider recurrences of the form

$$\left|{n \atop k}\right| = f_1(n,k)\left|{n-1 \atop k}\right| + f_2(n,k)\left|{n-1 \atop k-1}\right| + [n=k=0], \text{ for } n,k \geq 0, \tag{2}$$

assuming that $\left|{n \atop k}\right| = 0$ when $n < 0$ or $k < 0$. Equation (1) is the case $f_1 = f_2 = 1$.

To derive the solution to Equation (1), we need to establish some properties of recurrence relations of the form of Equation (2). First, it is fairly clear from the recurrence that the only nonzero elements of $\left|{n \atop k}\right|$ occur in the triangle $n \geq k \geq 0$. The second has to do with how changing the f_1 and f_2 functions changes the solution to Equation (2).

Theorem 1. *Suppose*

$$\left|{n \atop k}\right| = f_1(n,k)\left|{n-1 \atop k}\right| + f_2(n,k)\left|{n-1 \atop k-1}\right| + [n=k=0]$$

Math. Mag. **89** (2016) 192–195. doi:10.4169/math.mag.89.3.192.
MSC: Primary 05A10.

and

$$\left\|{n \atop k}\right\| = h(n)g_1(n-k)f_1(n,k)\left\|{n-1 \atop k}\right\| + h(n)g_2(k)f_2(n,k)\left\|{n-1 \atop k-1}\right\| + [n=k=0].$$

Then the solutions $\left|{n \atop k}\right|$ *and* $\left\|{n \atop k}\right\|$ *are related by*

$$\left\|{n \atop k}\right\| = \left(\prod_{j=1}^{n} h(j)\right)\left(\prod_{j=1}^{n-k} g_1(j)\right)\left(\prod_{j=1}^{k} g_2(j)\right)\left|{n \atop k}\right|. \tag{3}$$

Proof. The theorem is clearly true in the case $n = k = 0$. Otherwise,

$$\begin{aligned}
&h(n)g_1(n-k)f_1(n,k)\left(\prod_{j=1}^{n-1} h(j)\right)\left(\prod_{j=1}^{n-1-k} g_1(j)\right)\left(\prod_{j=1}^{k} g_2(j)\right)\left|{n-1 \atop k}\right| \\
&\quad + h(n)g_2(k)f_2(n,k)\left(\prod_{j=1}^{n-1} h(j)\right)\left(\prod_{j=1}^{n-k} g_1(j)\right)\left(\prod_{j=1}^{k-1} g_2(j)\right)\left|{n-1 \atop k-1}\right| \\
&= \left(\prod_{j=1}^{n} h(j)\right)\left(\prod_{j=1}^{n-k} g_1(j)\right)\left(\prod_{j=1}^{k} g_2(j)\right) f_1(n,k)\left|{n-1 \atop k}\right| \\
&\quad + \left(\prod_{j=1}^{n} h(j)\right)\left(\prod_{j=1}^{n-k} g_1(j)\right)\left(\prod_{j=1}^{k} g_2(j)\right) f_2(n,k)\left|{n-1 \atop k-1}\right| \\
&= \left(\prod_{j=1}^{n} h(j)\right)\left(\prod_{j=1}^{n-k} g_1(j)\right)\left(\prod_{j=1}^{k} g_2(j)\right)\left|{n \atop k}\right|.
\end{aligned}$$

Since the right-hand side of Equation (3) satisfies the recurrence for $\left\|{n \atop k}\right\|$, the theorem holds.

Theorem 1 tells us three things:

1. If we multiply f_1 and f_2 by the same function $h(n)$, the solution to the recurrence is multiplied by a factor of $\prod_{j=1}^{n} h(j)$.
2. If we multiply f_1 by the function $g_1(n-k)$, the solution to the recurrence is multiplied by a factor of $\prod_{j=1}^{n-k} g_1(j)$.
3. If we multiply f_2 by the function $g_2(k)$, the solution to the recurrence is multiplied by a factor of $\prod_{j=1}^{k} g_2(j)$.

Since we can use Theorem 1 to create solutions to recurrences of the form (2) from known solutions, a natural question to ask is this: What are the simplest (in some sense) functions f_1 and f_2 that give rise to $\left|{n \atop k}\right| = 1$ for $n \geq k \geq 0$ (i.e., yield a triangle of 1's)?

Let's start with the boundaries of the triangle. Since $\left|{n \atop -1}\right| = 0$, when generating the column $\left|{n \atop 0}\right| = 1$ the recurrence simplifies to $\left|{n \atop 0}\right| = \left|{n-1 \atop 0}\right|$. This implies that $f_1(n,0) = 1$. Similarly, since $\left|{n-1 \atop n}\right| = 0$, to generate the diagonal $\left|{n \atop n}\right| = 1$, the recurrence simplifies to $\left|{n \atop n}\right| = \left|{n-1 \atop n-1}\right|$, and thus $f_2(n,n) = 1$. Off of the boundaries, substituting 1's into $\left|{n \atop k}\right|$, $\left|{n-1 \atop k}\right|$, and $\left|{n-1 \atop k-1}\right|$, we get that $f_1(n,k) + f_2(n,k) = 1$. This rules out constant solutions for f_1 and f_2. The simplest solutions that depend on n and k and that yield $f_2(n,n) = 1$

are probably $f_2(n,k) = \frac{n}{k}$ and $f_2(n,k) = \frac{k}{n}$. The former leads to $f_1(n,k) = 1 - \frac{n}{k}$, which is undefined when $k = 0$ and thus is ruled out. However, the latter leads to $f_1(n,k) = 1 - \frac{k}{n}$, which satisfies the other boundary condition $f_1(n,0) = 1$. We thus have our third property of recurrences of the form of Equation (2), which we might as well call a theorem:

Theorem 2. *If $f_1(n,k) = 1 - \frac{k}{n}$ and $f_2(n,k) = \frac{k}{n}$, then the solution to Equation* (2) *is $\left|{n \atop k}\right| = 1$ for $n \geq k \geq 0$.*

(The functions f_1 and f_2 in Theorem 2 have the additional nice property that $f_1(n,n) = 0$ and $f_2(n,0) = 0$. Thus, even if the definition did not force the values off of the triangle $n \geq k \geq 0$ to be zero, these off-values would still not affect the values inside the triangle.)

With Theorems 1 and 2 in place, the proof of our main result is straightforward.

Theorem 3. *The solution to*

$$\left|{n \atop k}\right| = \left|{n-1 \atop k}\right| + \left|{n-1 \atop k-1}\right| + [n = k = 0]$$

is

$$\left|{n \atop k}\right| = \frac{n!}{k!\,(n-k)!}.$$

Proof. Starting with $f_1(n,k) = 1 - \frac{k}{n}$ and $f_2(n,k) = \frac{k}{n}$ from Theorem 2, let's find functions h, g_1, and g_2 such that $h(n)g_1(n-k)f_1(n,k) = h(n)g_2(k)f_2(n,k) = 1$. We need to clear the denominators of f_1 and f_2, and so we need $h(n) = n$. This leaves $g_1(n-k)(n-k) = 1$, and thus $g_1(n-k) = \frac{1}{n-k}$. Similarly, we have $g_2(k)k = 1$, and thus $g_2(k) = \frac{1}{k}$. By Theorem 1, we thus have that the solution is

$$\left|{n \atop k}\right| = \left(\prod_{j=1}^{n} j\right)\left(\prod_{j=1}^{n-k} \frac{1}{j}\right)\left(\prod_{j=1}^{k} \frac{1}{j}\right) = \frac{n!}{(n-k)!\,k!}.$$

Recurrence relations of the form of Equation (2) have generally been difficult to solve, even though many important named numbers are special cases. (Besides the binomial coefficients, different forms of f_1 and f_2 generate both kinds of Stirling and associated Stirling numbers, the Lah numbers, the Gaussian coefficients, the Eulerian numbers, and second-order Eulerian numbers. See Konvalina [**4**] for a unified combinatorial interpretation of some of these numbers.) In fact, Research Problem 6.94 in Graham, Knuth, and Patashnik's *Concrete Mathematics* [**3**] says

> Develop a general theory of the solutions to the two-parameter recurrence
>
> $$\left|{n \atop k}\right| = (\alpha n + \beta k + \gamma)\left|{n-1 \atop k}\right| + (\alpha' n + \beta' k + \gamma')\left|{n-1 \atop k-1}\right| + [n = k = 0],$$
>
> for $n, k \geq 0$, (4)
>
> assuming that $\left|{n \atop k}\right| = 0$ when $n < 0$ or $k < 0$. What special values ($\alpha, \beta, \gamma, \alpha', \beta', \gamma'$) yield "fundamental solutions" in terms of which the general solution can be expressed?

Much recent progress has been made on this problem, however. In previous work [**6**], we use our Theorem 1 and some similar results to find explicit solutions to several cases of Equation (4). In addition, Mansour and Shattuck [**5**] give combinatorial interpretations of several cases of Equation (4). Finally, a recent article by Barbero, Salas, and Villaseñor [**2**] gives a complete solution to Equation (4) in terms of generating functions.

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Summary. We give a new, direct argument that the solution to the binomial recurrence is the binomial coefficient. Our argument does not assume that the solution is known in advance nor does it rely on anything other than basic properties of two-variable triangular recurrence relations.

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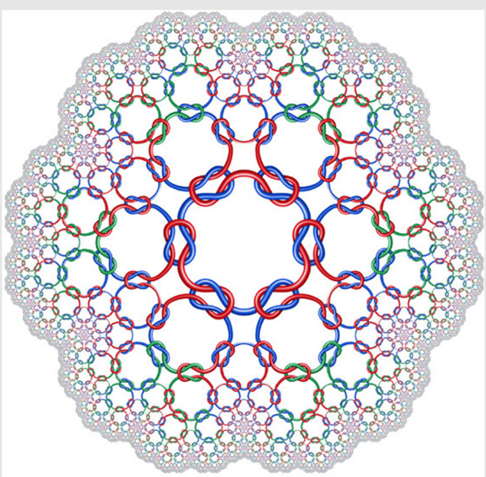
Artist Spotlight
Robert Fathauer



Square Knots Stretching to Infinity, Robert Fathauer; 14 in. × 14 in. limited edition of 50 digital prints, 2007. This print depicts a fractal link design where each strand has four lobes and connects to other strands via square knots. Three colors are required to avoid knotted strands having the same color. The design was created by decorating the individual tiles in a fractal tiling with knot-like graphics.

See interview on page 220.