The Chu–Vandermonde Identity via Leibniz's Identity for Derivatives

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While the product rule $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ rightly gets a lot of play time in a course on differentiation, Leibniz's identity

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x),$$

which generalizes the product rule to higher-order derivatives, rarely makes it off the bench. In this note we describe an application of Leibniz's identity that could be used as an advanced example or a project in a calculus course or a combinatorics course.

Specifically, we use Leibniz's identity to prove the Chu–Vandermonde identity for the binomial coefficients.

Theorem 1 (Chu–Vandermonde Identity). *For $n, m \in \mathbb{R}$ and $r \in \mathbb{N}_0$,*

$$\sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k} = \binom{n+m}{r}.$$

When $n, m \in \mathbb{N}_0$, the identity is generally known as *Vandermonde's convolution* and has a nice combinatorial proof [**1**, p. 66–67].

The Chu–Vandermonde identity, though, is valid when n and m are real numbers, not just natural numbers. To even understand Chu–Vandermonde, we have to extend the binomial coefficient $\binom{n}{k}$ to the case where n is real. The usual way to do that is via *falling factorial powers*:

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

This definition agrees with the usual factorial definition of the binomial coefficients when $n \in \mathbb{N}_0$.

Before we get to the proof of Chu–Vandermonde, one might ask, either pedagogically or mathematically, why we would want to extend Vandermonde's convolution to real n and m. For one, it introduces the idea of the generalized binomial coefficient. Calculus students are going to see generalized binomial coefficients later when they encounter Newton's binomial series for real n:

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k, \text{ valid for } -1 < x < 1.$$

In addition, a discussion of the generalized binomial coefficient with Chu–Vandermonde as an example can lead to a discussion of the gamma function as a generalization of the factorial.

There are also combinatorial identities that can be easily proved using Chu–Vandermonde but for which Vandermonde's convolution alone is not sufficient. For

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example, with the identity $\binom{-1/2}{n}(-4)^n = \binom{2n}{n}$ and Chu–Vandermonde, it becomes quite easy to prove

$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k} = 4^n$$

[**2**, p. 187]. Combinatorial proofs of this identity are rather more involved [**4**]. Proving Chu–Vandermonde via Leibniz's identity also serves as a more sophisticated use of calculus in combinatorics than the typical differentiate-the-binomial-theorem examples. Finally, for more advanced students, Chu–Vandermonde can be used in a discussion of hypergeometric functions, as it is a special case of Gauss's hypergeometric theorem [**3**, p. 133].

We now give the promised proof.

Proof. Let $f(x) = x^n$ and $g(x) = x^m$ for $n, m \in \mathbb{R}$. Differentiating a power function like x^n a total of r times yields $n(n-1)\cdots(n-r+1)x^{n-r} = n^{\underline{r}}x^{n-r}$. Thus $(fg)^{(r)}(x) = \frac{d^r}{dx^r}(x^{n+m}) = (n+m)^{\underline{r}}x^{n+m-r}$. Applying Leibniz's identity gives

$$\begin{aligned}(n+m)^{\underline{r}}x^{n+m-r} &= \sum_{k=0}^{r}\binom{r}{k}n^{\underline{k}}x^{n-k}m^{\underline{r-k}}x^{m-r+k}\\ &= \sum_{k=0}^{r}\frac{r!}{(r-k)!\,k!}n^{\underline{k}}m^{\underline{r-k}}x^{n+m-r},\end{aligned}$$

as $r, k \in \mathbb{N}_0$. Dividing both sides by $r!$ and regrouping, we have

$$\frac{(n+m)^{\underline{r}}}{r!}x^{n+m-r} = \sum_{k=0}^{r}\frac{n^{\underline{k}}}{k!}\,\frac{m^{\underline{r-k}}}{(r-k)!}\,x^{n+m-r},$$

or, using the generalized binomial coefficient notation,

$$\binom{n+m}{r}x^{n+m-r} = \left(\sum_{k=0}^{r}\binom{n}{k}\binom{m}{r-k}\right)x^{n+m-r}.$$

In order for these two power functions to be equal, their coefficients must be equal. ■

Summary. We give a proof of the Chu–Vandermonde identity using only Leibniz's identity for derivatives.

References

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