The Lah Numbers and the nth Derivative of $e^{1/x}$

Siad Daboul
53111 Bonn
Germany
s6sidabo@uni-bonn.de

Jan Mangaldan
Parañaque 1709
Philippines
pleasureoffiguring@gmail.com

Michael Z. Spivey
University of Puget Sound
Tacoma, Washington 98416
USA
mspivey@pugetsound.edu

Peter Taylor
Valencia 46019
Spain
pjt33@cantab.net

What do you get when you take the derivative of $e^{1/x}$ n times? The result must be of the form $e^{1/x}$ times a sum of various powers of x^{-1}, but what, exactly, are the coefficients of those powers?

In this article we show that these coefficients are the *Lah numbers*, a triangle of integers whose best-known applications are in combinatorics and finite difference calculus. We give five proofs, using four different properties of the Lah numbers. In the process we take a tour through several areas of mathematics, seeing the binomial coefficients, Faà di Bruno's formula, set partitions, Maclaurin series, factorial powers, the Poisson probability distribution, and hypergeometric functions.

The first proof: the direct approach

First we step through the process whereby one might construct a conjecture for the nth derivative of $e^{1/x}$. The natural thing to do is to calculate the derivative for several small values of n and look for a pattern. This leads to the following.

n	$D^n(e^{1/x})$
1	$-e^{1/x}x^{-2}$
2	$e^{1/x}\left(2x^{-3}+x^{-4}\right)$
3	$-e^{1/x}\left(6x^{-4}+6x^{-5}+x^{-6}\right)$
4	$e^{1/x}\left(24x^{-5}+36x^{-6}+12x^{-7}+x^{-8}\right)$
5	$-e^{1/x}\left(120x^{-6}+240x^{-7}+120x^{-8}+20x^{-9}+x^{-10}\right)$

The most famous triangle of numbers, of course, is the triangle of binomial coefficients $\binom{n}{k}$, which starts like so.

n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

Ignoring signs, the coefficients in the table of derivatives of $e^{1/x}$ look suspiciously like multiples of the binomial coefficients. Following up with this conjecture, and letting $c_{n,k}$ be the coefficient of x^{-n-k} in the nth derivative of $e^{1/x}$, we get this table.

n	$c_{n,0}/\binom{n}{0}$	$c_{n,1}/\binom{n}{1}$	$c_{n,2}/\binom{n}{2}$	$c_{n,3}/\binom{n}{3}$	$c_{n,4}/\binom{n}{4}$	$c_{n,5}/\binom{n}{5}$
0	1					
1	0	1				
2	0	1	1			
3	0	2	2	1		
4	0	6	6	3	1	
5	0	24	24	12	4	1

These numbers also look like they might be multiples of the binomial coefficients, just shifted in n and k. Conjecturing that these numbers are multiples of $\binom{n-1}{k-1}$, we have the following.

n	$\frac{c_{n,0}}{\binom{n}{0}}$	$\frac{c_{n,1}}{\binom{n}{1}\binom{n-1}{0}}$	$\frac{c_{n,2}}{\binom{n}{2}\binom{n-1}{1}}$	$\frac{c_{n,3}}{\binom{n}{3}\binom{n-1}{2}}$	$\frac{c_{n,4}}{\binom{n}{4}\binom{n-1}{3}}$	$\frac{c_{n,5}}{\binom{n}{5}\binom{n-1}{4}}$
0	1					
1	0	1				
2	0	1	1			
3	0	2	1	1		
4	0	6	2	1	1	
5	0	24	6	2	1	1

This triangle of numbers we immediately recognize to be the factorials, with (except for column $k = 0$) the number in entry n, k as $(n-k)!$. This pattern seems too regular to be a coincidence, and, in fact, the pattern continues. We have this theorem.

THEOREM 1.

$$\frac{d^n}{dx^n}\left(e^{1/x}\right) = (-1)^n e^{1/x} \sum_{k=1}^{n} \binom{n}{k}\binom{n-1}{k-1}(n-k)!x^{-n-k}.$$

Proof. This looks like a prime candidate for induction. We have already shown that the expression holds for $n = 1, 2, 3, 4$, and 5. Then

$$\begin{aligned}\frac{d^{n+1}}{dx^{n+1}}\left(e^{1/x}\right) &= \frac{d}{dx}\left((-1)^n e^{1/x} \sum_{k=1}^{n} \binom{n}{k}\binom{n-1}{k-1}(n-k)!x^{-n-k}\right)\\ &= (-1)^{n+1} e^{1/x}\left(\sum_{k=1}^{n} \binom{n}{k}\binom{n-1}{k-1}(n-k)!x^{-n-k-2}\right.\\ &\qquad \left.+ \sum_{k=1}^{n} \binom{n}{k}\binom{n-1}{k-1}(n-k)!(n+k)x^{-n-k-1}\right).\end{aligned}$$

Shifting indices on the first sum and using the fact that $\binom{n}{-1} = \binom{n}{n+1} = 0$, we can combine the two sums into one:

$$\sum_{k=1}^{n+1}\left(\binom{n}{k-1}\binom{n-1}{k-2}(n-k+1)! + \binom{n}{k}\binom{n-1}{k-1}(n-k)!(n+k)\right)x^{-n-k-1}.$$

Focusing now on the binomial expression in the summand, we have

$$\begin{aligned}
&\binom{n}{k-1}\binom{n-1}{k-2}(n-k+1)! + \binom{n}{k}\binom{n-1}{k-1}(n-k)!(n+k) \\
&= \frac{n!\,(n-1)!}{(k-1)!\,(k-2)!\,(n+1-k)!} + \frac{n!\,(n-1)!\,(n+k)}{k!\,(k-1)!\,(n-k)!} \\
&= \frac{(n+1)!\,n!}{k!\,(k-1)!\,(n+1-k)!}\left(\frac{k(k-1)}{n(n+1)} + \frac{(n+k)(n+1-k)}{n(n+1)}\right) \\
&= \binom{n+1}{k}\binom{n}{k-1}(n+1-k)!\left(\frac{k^2-k+n^2-k^2+n+k}{n(n+1)}\right) \\
&= \binom{n+1}{k}\binom{n}{k-1}(n+1-k)!,
\end{aligned}$$

completing the proof.

The Lah numbers

The numbers, $L(n,k) = \binom{n}{k}\binom{n-1}{k-1}(n-k)!$, have been studied in other contexts. The Slovenian mathematician Ivo Lah first investigated their properties in a pair of papers [7, 8] from the 1950s, and as a result they are called the *Lah numbers.* They appear as sequence A008297 in the On-line Encyclopedia of Integer Sequences [10]. The following table contains the first several rows of the Lah number triangle.

Lah numbers

$n\backslash k$	1	2	3	4	5	6
1	1					
2	2	1				
3	6	6	1			
4	24	36	12	1		
5	120	240	120	20	1	
6	720	1800	1200	300	30	1

Besides their representation $\binom{n}{k}\binom{n-1}{k-1}(n-k)!$ in terms of binomial coefficients, some of the better-known properties of the Lah numbers $L(n,k)$ are given in the next theorem.

Theorem 2.

1. The Lah numbers satisfy a nice triangular recurrence relation: $L(n+1,k) = (n+k)L(n,k) + L(n,k-1)$, with $L(n,0) = 0$, $L(n,k) = 0$ for $n < k$, and $L(1,1) = 1$.

2. The Lah numbers are the coefficients used to convert rising factorial powers to falling factorial powers: $x^{\overline{n}} = \sum_{k=1}^{n} L(n,k)x^{\underline{k}}$, where $x^{\overline{n}} = x(x+1)\cdots(x+n-1)$ and $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$. (This was the property that Lah was originally studying.)

3. The Lah number $L(n,k)$ counts the number of ways a set of n elements can be partitioned into k nonempty tuples.

4. The Lah numbers can be represented in terms of Stirling numbers: $L(n,k) = \sum_{j=k}^{n} \left[{n \atop j}\right]\left\{{j \atop k}\right\}$, where $\left[{n \atop j}\right]$ and $\left\{{j \atop k}\right\}$ are Stirling numbers of the first and second kinds, respectively.

We use the first three of these properties in subsequent proofs that the Lah numbers are the coefficients in the nth derivative of $e^{1/x}$.

Proof. We take $L(n,k) = \binom{n}{k}\binom{n-1}{k-1}(n-k)!$ as the definition of the Lah numbers.

Property 3. To construct k nonempty tuples, first choose a permutation of n elements. This can be done in $n!$ ways. Then choose $k-1$ of the $n-1$ possible cut points in the permutation to create k nonempty tuples. This can be done in $\binom{n-1}{k-1}$ ways. However, since there are $k!$ ways to order the tuples, there are $k!$ permutations that create the same set of k nonempty tuples. Thus the number of ways a set of n elements can be partitioned into k nonempty tuples is $\frac{n!}{k!}\binom{n-1}{k-1} = \binom{n}{k}\binom{n-1}{k-1}(n-k)! = L(n,k)$.

Property 1 from Property 3. One way to count the number of ways of constructing k tuples from $n+1$ elements is to condition on element $n+1$. The number of ways to construct k tuples from $n+1$ elements in which element $n+1$ is in a tuple by itself is $L(n,k-1)$. Otherwise, element $n+1$ could be placed in front of any of n elements in k already-existing tuples, in n ways, or at the end of one of the tuples, in k ways. There are $(n+k)L(n,k)$ ways to do this. Clearly, it is impossible to construct 0 nonempty tuples from n elements or if there are more tuples than elements, and there is one way to construct a nonempty tuple from one element.

Property 2 from Property 3. First, assume that x is a positive integer. We show that the two sides of the equation in Property 2 count the number of ways to construct x (not necessarily nonempty) ordered tuples from n elements. For the left side, construct the tuples by placing elements one-by-one in the x possible tuples. If there are j elements already placed, then element $j+1$ can be placed in front of any of the already-placed elements, in j ways, or it can be placed at the end of one of the x tuples, in x ways, for a total of $x+j$ possible placements. Thus the number of ways to construct the x tuples is $x(x+1)\cdots(x+n-1) = x^{\overline{n}}$. For the right side, condition on the number of nonempty tuples. There are $L(n,k)$ ways to form the k nonempty tuples from the n elements, and then there are $x(x-1)\cdots(x-k+1) = x^{\underline{k}}$ ways to order the tuples (including the empty ones). Summing up over the possible values of k yields $\sum_{k=1}^{n} L(n,k)x^{\underline{k}}$, which then must also be the total number of ways to construct x ordered tuples from n elements. This proves Property 2 for x a positive integer. However, the equation in Property 2 is a polynomial of degree n. Since it takes $n+1$ values to specify such a polynomial, and we have just shown that this equation holds for an infinite number of values of x, Property 2 must hold for all real values of x.

Property 4 from Property 2. Stirling numbers of the first kind $\left[{n \atop k}\right]$ are the coefficients when converting from rising powers to ordinary powers, and Stirling numbers of the second kind $\left\{{n \atop k}\right\}$ are the coefficients when converting from ordinary powers to falling powers [5, p. 264]. Thus we have $x^{\overline{n}} = \sum_{j=1}^{n} \left[{n \atop j}\right] x^j = \sum_{j=1}^{n} \left[{n \atop j}\right] \sum_{k=1}^{j} \left\{{j \atop k}\right\} x^{\underline{k}} = \sum_{k=1}^{n} \sum_{j=k}^{n} \left[{n \atop j}\right] \left\{{j \atop k}\right\} x^{\underline{k}}$, and so $L(n,k) = \sum_{j=k}^{n} \left[{n \atop j}\right] \left\{{j \atop k}\right\}$. □

(See also Petkovšek and Pisanski [11] for proofs of Properties 1, 2, and 3.)

We now present four more proofs that $\frac{d^n}{dx^n}\left(e^{1/x}\right) = (-1)^n e^{-1/x} \sum_{k=1}^{n} L(n,k) x^{-n-k}$. The first three use Properties 1, 2, and 3, respectively, of the Lah numbers. The last proof uses Kummer's confluent hypergeometric function transformation as well as the representation of the Lah numbers in terms of binomial coefficients in Theorem 1.

The second proof: use the recurrence relation

The recurrence relation given in Property 1 should greatly simplify our first induction proof. For the induction step, we have

$$\begin{aligned}
\frac{d^{n+1}}{dx^{n+1}}\left(e^{1/x}\right) &= \frac{d}{dx}\left((-1)^n e^{1/x} \sum_{k=1}^{n} L(n,k) x^{-n-k}\right) \\
&= (-1)^{n+1} e^{1/x} \left(\sum_{k=1}^{n} L(n,k) x^{-n-k-2} + \sum_{k=1}^{n} L(n,k)(n+k) x^{-n-k-1}\right) \\
&= (-1)^{n+1} e^{1/x} \sum_{k=1}^{n+1} \left(L(n,k-1) + (n+k)L(n,k)\right) x^{-n-k-1} \\
&= (-1)^{n+1} e^{1/x} \sum_{k=1}^{n+1} L(n+1,k) x^{-(n+1)-k}.
\end{aligned}$$

(In fact, we can now see that our first proof was essentially a proof of Property 1 of the Lah numbers.)

The third proof: factorial moments of the Poisson distribution

Our third proof uses the Poisson probability distribution and the Maclaurin series for e^x. The Poisson distribution is often used to model the number of occurrences of some event over a fixed period of time, such as the number of arrivals to a fast-food restaurant in an hour or the number of salmon that swim by a particular point on a river in a day. The probability mass function for a Poisson distribution is $f(j) = \lambda^j e^{-\lambda}/j!$, for $j = 0, 1, \ldots$. Here λ is a rate parameter; for example, λ might represent the expected number of occurrences in the given time interval.

To begin our third proof, consider the Maclaurin series for e^x:

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}.$$

Substituting x^{-1} for x and differentiating n times yields

$$\frac{d^n}{dx^n}\left(e^{1/x}\right) = (-1)^n x^{-n} \sum_{j=0}^{\infty} \frac{j^{\overline{n}} x^{-j}}{j!}.$$

The summand is similar to the probability mass function of a Poisson distribution with $\lambda = x^{-1}$ for the rate parameter. In fact, with an $e^{-1/x}$ in the summand the infinite series is exactly the nth rising factorial moment $E\left[X^{\overline{n}}\right]$. While the rising factorial moments of a Poisson distribution do not have a known simple expression, it is known that the falling factorial moments do, and this is easily proved. If Y is Poisson, then

$$\begin{aligned} E[Y^{\underline{n}}] &= \sum_{j=0}^{\infty} \frac{j^{\underline{n}} \lambda^j e^{-\lambda}}{j!} = e^{-\lambda} \sum_{j=n}^{\infty} \frac{j^{\underline{n}} \lambda^j}{j!} = e^{-\lambda} \sum_{j=n}^{\infty} \frac{j! \lambda^j}{(j-n)! j!} = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^{i+n}}{i!} = e^{-\lambda} \lambda^n e^{\lambda} \\ &= \lambda^n. \end{aligned}$$

Given this, it makes sense to convert rising powers to falling powers. Doing so, while using Property 2 of the Lah numbers to make the conversion, yields

$$\begin{aligned}
\frac{d^n}{dx^n}\left(e^{1/x}\right) &= (-1)^n x^{-n} \sum_{j=0}^{\infty} \sum_{k=1}^{n} L(n,k) \frac{j^{\underline{k}} x^{-j}}{j!} \\
&= (-1)^n x^{-n} e^{1/x} \sum_{k=1}^{n} L(n,k) E[X^{\underline{k}}] \\
&= (-1)^n e^{1/x} \sum_{k=1}^{n} L(n,k) x^{-n-k},
\end{aligned}$$

where X is Poisson (with rate parameter $\lambda = x^{-1}$) and $E[X^{\underline{k}}] = x^{-k}$.

The fourth proof: Faà di Bruno's formula

Our fourth proof uses a formula named after Faà di Bruno for the nth derivative of a composite function, as well as the combinatorial interpretation of the Lah numbers (Property 3). Faà di Bruno's papers [2] and [3] mentioning his formula date from the mid-1800s. However, neither contains a proof of the formula (merely its statement), and Faà di Bruno was not actually the first to state the formula nor to prove it! Readers interested in learning more should consult Warren Johnson's paper [6] "The curious history of Faà di Bruno's formula."

Faà di Bruno's formula can be represented in multiple forms [6]; for instance, there is a combinatorial version, a version that uses Bell polynomials, and a determinantal version. The most useful for our purposes is the combinatorial one:

$$\frac{d^n}{dx^n} f(g(x)) = \sum f^{(k)}(g(x))(g'(x))^{s_1}(g''(x))^{s_2} \cdots (g^{(n-k+1)}(x))^{s_{n-k+1}}. \tag{1}$$

Here the sum is over all partitions of $\{1, 2, \ldots, n\}$ into nonempty sets, where k is the number of sets in the partition and s_i is the number of sets with exactly i elements.

Why, exactly, does the complicated expression on the right of Equation (1) give the nth derivative of the composite function $f(g(x))$? An induction proof can be given using the following ideas. Each partition of $\{1, 2, \ldots, n+1\}$ can be formed in exactly one way by adding $n+1$ to a partition of $\{1, 2, \ldots, n\}$. If we add $\{n+1\}$ as a singleton set, that increases the total number of sets by 1 and the number of sets containing one element by 1. This corresponds to differentiating $f^{(k)}(g(x))$ to get $f^{(k+1)}(g(x))g'(x)$. If we add $n+1$ to an already-existing set containing i elements, that decreases the number of sets containing i elements by 1 and increases the number of sets containing $i+1$ elements by 1. With s_i sets containing i elements, this corresponds to differentiating $(g^{(i)}(x))^{s_i}$ to get $(g^{(i)}(x))^{s_i-1}g^{(i+1)}(x)$. Thus differentiating the right-hand side of Equation (1) corresponds to all the different ways to obtain a partition of $\{1, 2, \ldots, n+1\}$ by adding $n+1$ to a partition of $\{1, 2, \ldots, n\}$.

Since $f(x) = e^x$, and $g(x) = 1/x$, Faà di Bruno's formula tells us that

$$\frac{d^n}{dx^n}\left(e^{1/x}\right) = \sum e^{1/x}(-x^{-2})^{s_1}(2x^{-3})^{s_2}\cdots((-1)^{n-k+1}(n-k+1)!\,x^{-n+k-2})^{s_{n-k+1}}.$$

This expression looks fairly complicated, but it simplifies nicely. Factor out $e^{1/x}$ and then regroup the rest by powers of -1, powers of x, and factorials, so that the summand is the product of the following three factors:

1. $(-1)^{s_1+2s_2+\cdots+(n-k+1)s_{n-k+1}}$,
2. $x^{-(2s_1+3s_2+\cdots+(n-k+2)s_{n-k+1})}$, and
3. $(1!)^{s_1}(2!)^{s_2}\cdots((n-k+1)!)^{s_{n-k+1}}$.

For a given partition P_k of n elements into k nonempty sets, $s_1 + 2s_2 + \cdots + (n-k+1)s_{n-k+1}$ is the total number of elements in P_k, and so the first factor simplifies to $(-1)^n$. Similarly, $s_1 + s_2 + \cdots + s_{n-k+1}$ is the total number of sets in P_k, and so the second factor simplifies to x^{-n-k}. Finally, each set of size i in P_k can have its elements permuted in $i!$ ways to create a tuple, and so the third factor is the number of partitions of $\{1, 2, \ldots, n\}$ into nonempty tuples that can be formed from the partition P_k of $\{1, 2, \ldots, n\}$ into nonempty sets. Therefore,

$$\begin{aligned}\frac{d^n}{dx^n}\left(e^{1/x}\right) &= (-1)^n e^{1/x}\sum_{k=1}^{n} x^{-n-k}\sum_{P_k}(1!)^{s_1}(2!)^{s_2}\cdots((n-k+1)!)^{s_{n-k+1}}\\ &= (-1)^n e^{1/x}\sum_{k=1}^{n} L(n,k)x^{-n-k}.\end{aligned}$$

The fifth proof: Kummer's hypergeometric transformation

Our last proof uses properties of hypergeometric functions – especially Kummer's hypergeometric transformation – in addition to the binomial coefficient expression for the Lah numbers.

A general *hypergeometric function* is of the form

$$\sum_{k=0}^{\infty} \frac{a_1^{\overline{k}} \cdots a_m^{\overline{k}}}{b_1^{\overline{k}} \cdots b_n^{\overline{k}}} \frac{z^k}{k!}.$$

As such it is a power series in z with the a_i's and b_i's as parameters. A perhaps surprisingly large number of functions can be expressed as hypergeometric series, often through their Taylor expansions. The study of hypergeometric series goes back at least to Euler, and Gauss proved some of their properties in his doctoral dissertation. Graham, Knuth, and Patashnik's *Concrete Mathematics* [5, Ch. 5] contains a nice introduction to hypergeometric functions.

The version of the hypergeometric series we need is the one in which m and n are both 1; i.e.,

$$\sum_{k=0}^{\infty} \frac{a^{\overline{k}}}{b^{\overline{k}}} \frac{z^k}{k!}.$$

This series is known as *Kummer's confluent hypergeometric function* and is denoted ${}_1F_1(a; b; z)$ [4, §13.1], $F\left({a \atop b} \mid z\right)$, or $M(a, b, z)$ [5, p. 206]. One of the most important properties of Kummer's confluent hypergeometric function is the transformation [5, Exercise 5.29]

$$M(a, b, z) = e^z M(b - a, b, -z). \tag{2}$$

For our final proof, as in the third proof, we start with the Maclaurin series for e^x. Substituting x^{-1} for x and differentiating n times yields

$$\frac{d^n}{dx^n}\left(e^{1/x}\right) = (-1)^n x^{-n} \sum_{k=0}^{\infty} \frac{k^{\overline{n}} x^{-k}}{k!}.$$

The sum can be expressed as a hypergeometric series:

$$\sum_{k=0}^{\infty} \frac{k^{\overline{n}} x^{-k}}{k!} = \sum_{k=1}^{\infty} \frac{(n+k-1)!\, x^{-k}}{k!\,(k-1)!} = x^{-1} \sum_{k=0}^{\infty} \frac{(n+k)!\, x^{-k}}{(k+1)!\, k!} = x^{-1} n! \sum_{k=0}^{\infty} \frac{(n+1)^{\overline{k}}\, x^{-k}}{2^{\overline{k}}\, k!}.$$

Thus the series is of the form of Kummer's confluent hypergeometric function $M(n + 1, 2; 1/x)$. Applying Kummer's transformation (2), we have

$$\begin{aligned}
\frac{d^n}{dx^n}\left(e^{1/x}\right) &= (-1)^n x^{-n-1} n! M(n+1, 2, 1/x) \\
&= (-1)^n x^{-n-1} n!\, e^{1/x} M(1-n, 2, -1/x) \\
&= (-1)^n x^{-n-1} n!\, e^{1/x} \sum_{k=0}^{\infty} \frac{(1-n)^{\overline{k}}\, (-x)^{-k}}{2^{\overline{k}}\, k!}.
\end{aligned}$$

Since $(1-n)^{\overline{k}} = 0$ for $k > n-1$, we obtain

$$\begin{aligned}
\frac{d^n}{dx^n}\left(e^{1/x}\right) &= (-1)^n x^{-n-1} n! e^{1/x} \sum_{k=0}^{n-1} \frac{(-1)^k (n-1)^{\underline{k}}\, (-1)^k x^{-k}}{(k+1)!\, k!} \\
&= (-1)^n x^{-n-1} e^{1/x} \sum_{k=0}^{n-1} \frac{n!\,(n-1)!\,x^{-k}}{(n-1-k)!\,(k+1)!\,k!} \\
&= (-1)^n x^{-n} e^{1/x} \sum_{k=1}^{n} \frac{n!\,(n-1)!\,x^{-k}}{(n-k)!\,k!\,(k-1)!} \\
&= (-1)^n e^{1/x} \sum_{k=1}^{n} \binom{n}{k}\binom{n-1}{k-1}(n-k)! x^{-n-k} \\
&= (-1)^n e^{1/x} \sum_{k=1}^{n} L(n,k) x^{-n-k}.
\end{aligned}$$

For a unified treatment on evaluating hypergeometric sums see Petkovšek, Wilf, and Zeilberger's text *A=B* [12].

Remark After this paper was accepted for publication the authors found Theorem 1 in Comtet's *Advanced Combinatorics* [1, Ch. III, Ex. 7, p. 158]. Comtet mentions more general results as well, including the cases $e^{\sqrt{x}}$ and e^{x^2}.

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