Visualizing Continued Fractions

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A *continued fraction* is a representation of a number as a series of nested fractions. For example, 3/7 has the following representation as a continued fraction:

$$\frac{3}{7} = 0 + \cfrac{1}{2 + \frac{1}{3}}.$$

Deeply nested continued fractions are difficult to typeset, and thankfully there is an alternate notation:

$$\frac{3}{7} = 0 + \frac{1}{2+}\frac{1}{3}.$$

Continued fractions have been a topic of mathematical study for quite some time. For example, Euler devotes a full chapter of his *Introductio in Analysin Infinitorum* [1] to continued fractions.

We give a method of representing the usual construction of finite continued fractions in terms of a grid. This provides a more visual way of viewing continued fractions than one gets with the standard treatments. We hope that this alternative way of representing continued fractions gives new insight into them; in fact, we describe an unsolved problem involving continued fractions in terms of our grid representation.

Although the usual treatment of continued fractions is not geometric there are some exceptions. One appears as part of Hancock's *Development of the Minkowski Geometry of Numbers* [2] (see Chapters VII and X); however, our representation is different from anything in this work. Another is Klein's geometric interpretation of the convergents of an infinite continued fraction (see Olds [3, pp. 77–79]); this is also different from what we are trying to do, as we are working only with finite continued fractions.

Simple, finite continued fractions. The representation of 3/7 given above is a *simple continued fraction* — one in which the numerator of each fraction in the representation is a 1. The simple continued fraction $b_1 + \frac{1}{b_2+}\frac{1}{b_3+}\cdots\frac{1}{b_n}$ is also written as $[b_1, b_2, \ldots, b_n]$ with no loss of information.

The process of expressing 3/7 as a simple continued fraction goes something like this. First, 3/7 is smaller than 1, so we represent it like so:

$$\frac{3}{7} = 0 + \frac{1}{\frac{7}{3}}.$$

Then, we write 7/3 as a mixed fraction:

$$\frac{3}{7} = 0 + \cfrac{1}{2 + \frac{1}{3}}.$$

Since 3 is an integer we may stop; otherwise, we would have continued expressing the denominator of the fraction most deeply nested as a mixed fraction, creating a fraction one level deeper in the process. The procedure ends, as this one does, when the denominator of the most deeply nested fraction is an integer.

Our geometric representation of the process of finding the simple continued fraction representation of a/b is as follows.

Algorithm 1 (Simple, finite continued fractions)

1. Input a and b from the fraction a/b.

2. Let $r_0 = a$, $r_1 = b$.

3. Start at position (r_1, r_0) on the grid.

4. Let $i = 1$.

5. While $r_i \neq 0$

   (a) Let $b_i = \lfloor r_{i-1}/r_i \rfloor$.

   (b) Take b_i steps of length r_i down from (r_i, r_{i-1}).

   (c) Let the new position be denoted (r_i, r_{i+1}). (This means $r_{i+1} = r_{i-1} \bmod r_i$.)

   (d) If $r_{i+1} \neq 0$ then reflect about the 45-degree diagonal to the new position (r_{i+1}, r_i).

   (e) Let $i = i + 1$.

6. Let $n = i - 1$.

7. Output $[b_1, b_2, b_3, \ldots, b_n]$.

The algorithm can perhaps be best understood via an example. The representation of 3/7 given above is generated from the algorithm via the series of steps $A \to B \to C \to D \to E \to F$ shown in Figure 1. Reflection about the 45-degree diagonal corresponds to inverting the fractional remainder, and b_i steps correspond to b_i being in the denominator of the level $i-1$ nested fraction, as can be seen in Figure 2.

The proof that Algorithm 1 generates a simple continued fraction representation of a/b is as follows.

Theorem 1. *Algorithm 1 gives the continued fraction representation* $\frac{a}{b} = [b_1, b_2, b_3, \ldots, b_n]$.

Proof. First, for any iteration i it is clear that $r_{i-1} = b_i r_i + r_{i+1}$. Second, we must have $r_{i+1} < r_i$ for all $i \geq 1$; thus, the algorithm must terminate. Moreover, since the algorithm terminates on completion of iteration n we have $r_{n+1} = 0$ and $r_i > 0$ for $i \leq n$. The position at the start of iteration n is (r_n, r_{n-1}). Thus $[b_n] = b_n = r_{n-1}/r_n$. Now, fix i, $1 \leq i < n$, and suppose $r_{j-1}/r_j = [b_j, b_{j+1}, \ldots, b_n]$ for each j, $i+1 \leq j \leq n$. Thus $r_{i-1}/r_i = b_i + r_{i+1}/r_i = b_i + \frac{1}{r_i/r_{i+1}}$. Since $r_i/r_{i+1} = [b_{i+1}, b_{i+2} \ldots, b_n]$ we have $r_{i-1}/r_i = [b_i, b_{i+1}, \ldots, b_n]$. □

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9										
8										
7			C							
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5										
4										
3	E						A,B			
2										
1			D							
0	F									
	1	2	3	4	5	6	7	8	9	10

Figure 1: Grid example for Algorithm 1

Position	Geometric Operation	Continued Fraction Thus Far
A	Starting point	
B	Zero steps of 7 units each	0
C	Reflect across diagonal	$0+\frac{1}{}$
D	Two steps of 3 units each	$0+\frac{1}{2}$
E	Reflect across diagonal	$0+\frac{1}{2+\frac{1}{}}$
F	Three steps of 1 unit each	$0+\frac{1}{2+\frac{1}{3}}$

Figure 2: Interpretation of grid example for Algorithm 1

Algorithm 1 is also essentially the Euclidean algorithm for finding the greatest common divisor of a and b represented on a grid; all we need is the additional statement $gcd(a,b) = r_{i-1}$ at the end [4, p. 229]. Moreover, it is well-known that the operations in the Euclidean algorithm are the same as those for producing the simple continued fraction representation of $\frac{a}{b}$ [3, pp. 16-17]. Our proof, then, is not perhaps strictly necessary, but we present it for completeness.

General finite continued fractions. Some continued fractions are not simple. For example, 3/7 also has the following representation as a continued fraction:

$$\frac{3}{7} = 1 - \frac{2}{4 - \frac{1}{2}}$$

General finite continued fractions, in which negative values and fractions with numerators other than 1 are allowed, can be obtained by modifying Algorithm 1 in the following manner.

Algorithm 2 (General finite continued fractions)

1. Input a and b from the fraction a/b.
2. Let $r_0 = a$, $r_1 = b$.
3. Start at position (r_1, r_0) on the grid.
4. Let $i = 1$.
5. While $r_i \neq 0$
   (a) Choose an integer b_i and take b_i steps of length r_i down from (r_i, r_{i-1}).
   (b) Let the new position be denoted (r_i, s_i).
   (c) If $s_i < 0$
      i. Then
         A. Reflect about the horizontal axis to the new position $(r_i, -s_i)$.
         B. Choose an a_{i+1} from the set of negative integers that divide s_i (a_{i+1} could be -1).
      ii. Else choose an a_{i+1} from the set of positive integers that divide s_i (a_{i+1} could be 1).
   (d) Take a single step of length $|s_i| - s_i/a_{i+1}$ down, ending at position $(r_i, s_i/a_{i+1})$.
   (e) Let $r_{i+1} = s_i/a_{i+1}$.
   (f) If $r_{i+1} \neq 0$ then reflect about the 45-degree diagonal to the new position (r_{i+1}, r_i).
   (g) Let $i = i + 1$.
6. Let $n = i - 1$.
7. Output $b_1 + \frac{a_2}{b_2+} \frac{a_3}{b_3+} \cdots \frac{a_n}{b_n}$.

A major difference between Algorithm 1 and Algorithm 2 is that Algorithm 2 has two situations in each iteration in which a choice is involved.

The representation of 3/7 given above and generated by Algorithm 2 is shown in Figure 3. The value of a_i is the numerator in the level $i - 1$ continued fraction, and reflection about the horizontal axis (with the resulting negative a_i) indicates a negative sign, as we can see in Figure 4.

We now prove the correctness of Algorithm 2.

Theorem 2. *If it terminates Algorithm 2 gives the continued fraction representation* $\frac{a}{b} = b_1 + \frac{a_2}{b_2+} \frac{a_3}{b_3+} \cdots \frac{a_n}{b_n}$.

Proof. Because of the choice of the b_i's and a_i's there is no guarantee that Algorithm 2 terminates. However, if it does it functions very much the same as does Algorithm 1. The exceptions in iteration i are a_{i+1} and the possible reflection about

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-1		F								
-2										
-3										
-4							B			
	1	2	3	4	5	6	7	8	9	10

Figure 3: Grid example for Algorithm 2

the horizontal axis; the latter, however, is captured by the sign of a_{i+1}. As in the proof of Algorithm 1, we have $r_{n-1}/r_n = b_n$. Now, fix i, $1 \le i < n$, and suppose $r_{j-1}/r_j = b_j + \frac{a_{j+1}}{b_{j+1}+}\frac{a_{j+2}}{b_{j+2}+}\cdots\frac{a_n}{b_n}$ for each j, $i+1 \le j \le n$. As in the proof of Algorithm 1 we have $r_{i+1}/r_i = \frac{1}{b_{i+1}+}\frac{a_{i+2}}{b_{i+2}+}\cdots\frac{a_n}{b_n}$. Thus $s_i/r_i = a_{i+1}r_{i+1}/r_i = \frac{a_{i+1}}{b_{i+1}+}\frac{a_{i+2}}{b_{i+2}+}\cdots\frac{a_n}{b_n}$, and $r_{i-1}/r_i = b_i + s_i/r_i = b_i + a_{i+1}r_{i+1}/r_i = b_i + \frac{a_{i+1}}{b_{i+1}+}\frac{a_{i+2}}{b_{i+2}+}\cdots\frac{a_n}{b_n}$. □

An unsolved problem. An open problem involving continued fractions can be expressed in terms of the grid representations given here. Zaremba's conjecture is that there is a constant B such that for any integer $b > 1$ there is an integer a, $0 < a < b$, such that a and b are relatively prime and the simple continued fraction representation $a/b = [0, b_2, \ldots, b_n]$ has $b_i \le B$ for all i, $2 \le i \le n$. [5, p. 395], [6, pp. 93–119]. Computational work indicates that $B = 5$ should suffice [7]. Since $b_i = \lfloor r_{i-1}/r_i \rfloor$ for $i \le n$, this means that, in terms of the grid representation described by Algorithm 1, Zaremba's conjecture is that there is a constant B such that for any integer $b > 1$ there is an integer a, $0 < a < b$, such that a and b are relatively prime and all grid points in the representation for a/b except for the final one lie in the cone in the first quadrant with boundaries $y = Bx$ and $y = x/B$. Figure 5 gives this set of feasible grid points, for $B = 5$.

References

[1] Leonard Euler. *Introductio in Analysin Infinitorum.* Lausanne, 1748.

Position	Geometric Operation	Continued Fraction Thus Far
A	Starting point	
B	One step of 7 units	1
C	Reflect across horizontal axis	$1-$
D	Step half of the distance from the horizontal axis	$1-2$
E	Reflect across diagonal	$1-\frac{2}{}$
F	Four steps of 2 units each	$1-\frac{2}{4}$
G	Reflect across horizontal axis	$1-\frac{2}{4-}$
H	Reflect across diagonal	$1-\frac{2}{4-\frac{1}{}}$
I	Two steps of 1 unit each	$1-\frac{2}{4-\frac{1}{2}}$

Figure 4: Interpretation of grid example for Algorithm 2

[2] Harris Hancock. *Development of the Minkowski Geometry of Numbers.* Macmillan, New York, 1939.

[3] C. D. Olds. *Continued Fractions.* Random House, New York, 1963.

[4] Kenneth H. Rosen, editor. *Handbook of Discrete and Combinatorial Mathematics.* CRC Press, Boca Raton, Florida, 2000.

[5] Richard K. Guy. *Unsolved Problems in Number Theory.* Springer, New York, 3rd edition, 2004.

[6] S. K. Zaremba. La méthode des “bos treillis” pour le calcul des intégrales multiples. In *Applications of Number Theory to Numerical Analysis.* Academic Press, 1972.

[7] T. W. Cusick. Zaremba’s conjecture and sums of the divisor function. *Mathematics of Computation*, 61(203):171–176, 1993.

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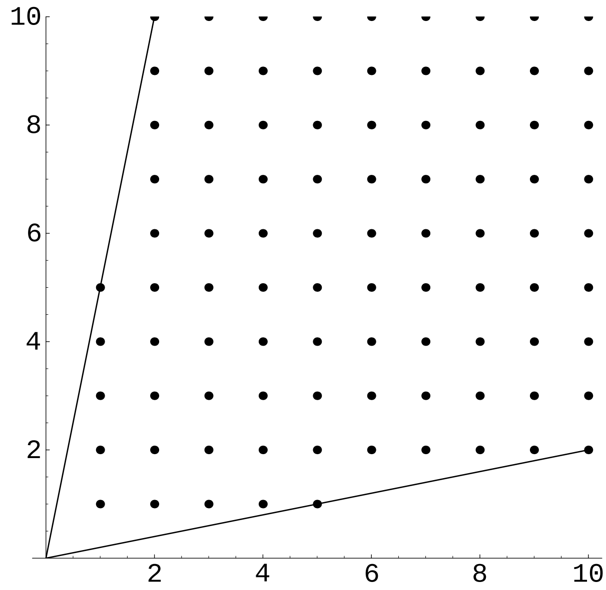

Figure 5: Grid representation of Zaremba's conjecture