A Combinatorial Proof for the Alternating Convolution of the Central Binomial Coefficients

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Abstract. We give a combinatorial proof of the identity for the alternating convolution of the central binomial coefficients. Our proof entails applying an involution to certain colored permutations and showing that only permutations containing cycles of even length remain.

The combinatorial identity

$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k} = 4^n \tag{1}$$

is fairly well known and is easy to prove using generating functions. Combinatorial proofs are more difficult, but there are some. Perhaps the most well known is a path-counting argument described by Sved [**7**]; De Angelis [**1**] gives another one using signed permutations. Chang and Xu [**2**] provide a probabilistic proof involving the expected value of a χ^2 random variable.

What about the alternating version of identity (1)? In this case, the identity is

$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k}(-1)^k = \begin{cases} 2^n\binom{n}{n/2}, & n \text{ even;} \\ 0, & n \text{ odd.} \end{cases} \tag{2}$$

Once again, the identity is easy to prove using generating functions. However, combinatorial proofs appear to be more difficult. There is a recent one due to Nagy [**3**], in which he shows bijectively that identity (2) is equivalent to a convolution of the Catalan numbers and the central binomial coefficients, which he then proves by a path-counting argument.

We present a new combinatorial proof of identity (2) involving colored permutations and even-length cycles. First, we need the identity

$$\sum_{k=1}^{n} \left[{n \atop k}\right] x^k = x^{\overline{n}}, \tag{3}$$

where $\left[{n \atop k}\right]$ is an unsigned Stirling number of the first kind, that is, the number of permutations on $[n]$ with exactly k cycles, and $x^{\overline{n}} = x(x+1)\cdots(x+n-1)$. Identity (3) is the well-known property that unsigned Stirling numbers of the first kind are used to convert rising factorial powers to ordinary powers. Since we wish to present a complete combinatorial argument, this identity needs a combinatorial proof as well. The following argument is known. (For a variant, see Stanley [**6**, pp. 34–35]).

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Proof. If x is a positive integer, both sides count the number of ways to construct cycle-colored permutations of $[n]$ when there are x colors. For the left side, we condition on the total number of cycles. There are $\left[{n \atop k}\right]$ ways to create a permutation on n elements with exactly k disjoint cycles, and then there are x^k ways to assign colors to those k cycles. For the right side, place the elements $1, 2, \ldots, n$ in order into colored cycles. If $1, 2, \ldots, k$ have already been placed, then $k + 1$ can be placed after any of the previous k elements, in k ways, or it can be used to start a new colored cycle, in x ways, for a total of $x + k$ ways to place $k + 1$. This proves the identity for all positive integers x. However, identity (3) entails a polynomial of degree n. Since we have proved this identity for an infinite number of values of x, it must hold for all real values of x. ■

Now to the proof of identity (2). Our proof involves combinatorial interpretations of the left and right sides of identity (2), followed by a bijection between the two interpretations. The interpretations are described in terms of probabilities, but the proofs still involve counting arguments.

Claim 1. *Select a permutation σ of $[n]$ uniformly at random. For each cycle w of σ, color w red with probability $1/2$; otherwise, color it blue. This creates a colored permutation. Then*

$$\binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{4^n}$$

is the probability that exactly k of the n elements of a randomly-chosen permutation σ are colored red.

Proof. There are $\binom{n}{k}$ ways to choose which k elements of a given permutation will be red and which $n - k$ elements will be blue. Given k particular elements of $[n]$, the number of ways those k elements can be expressed as the product of i disjoint cycles is $\left[{k \atop i}\right]$. Thus, the probability of choosing a permutation σ that has those k elements as the product of i disjoint cycles, all of which are colored red, and the remaining $n - k$ elements as the product of j disjoint cycles, all of which are colored blue, is $\left[{k \atop i}\right]\left[{n-k \atop j}\right]/(2^i 2^j n!)$. Summing up, the probability that exactly k of the n elements in a randomly chosen permutation are colored red is

$$\frac{\binom{n}{k}}{n!}\left(\sum_{i=1}^{k}\frac{\left[{k \atop i}\right]}{2^i}\right)\left(\sum_{j=1}^{n-k}\frac{\left[{n-k \atop j}\right]}{2^j}\right).$$

Using identity (3), the first sum is easily rewritten as

$$\sum_{i=1}^{k}\frac{\left[{k \atop i}\right]}{2^i} = \prod_{i=0}^{k-1}\left(\frac{1}{2}+i\right) = \frac{1(3)(5)\cdots(2k-1)}{2^k} = \frac{1(2)(3)\cdots(2k-1)(2k)}{2^k 2^k k!} = \frac{(2k)!}{4^k k!}.$$

Similarly, the second sum is

$$\sum_{j=1}^{n-k}\frac{\left[{n-k \atop j}\right]}{2^j} = \frac{(2n-2k)!}{4^{n-k}(n-k)!}.$$

Thus, the probability that exactly k of the n elements of a randomly chosen permutation are colored red is

$$\frac{\binom{n}{k}}{n!}\frac{(2k)!}{4^k k!}\frac{(2n-2k)!}{4^{n-k}(n-k)!} = \binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{4^n}. \qquad \blacksquare$$

As an aside, obviously

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{4^n} = 1,$$

which gives another combinatorial proof of identity (1). This is mentioned in Stanley's "Bijective Proof Problems" [**5**].

Claim 2. *Select a permutation σ of $[n]$ uniformly at random. The probability that σ contains only cycles of even length is*

$$\begin{cases} \frac{1}{2^n}\binom{n}{n/2}, & n \text{ even}; \\ 0, & n \text{ odd}. \end{cases}$$

Proof. Clearly, if n is odd, then there must be at least one odd cycle in σ. Assume, then, that n is even.

Construct a permutation σ of $[n]$ in stages. Start by placing 1 into a cycle. Then, at each stage, we can either (1) choose a new number to go at the end of the current cycle, or (2) close the current cycle and begin a new one with the smallest number not yet placed into a cycle.

Suppose that k elements have been placed into cycles, $k \geq 1$, and all closed cycles have even length. If k is odd, closing the current cycle would create an odd-length cycle. Thus, we must choose a new number to go at the end of the current cycle. This means that there are $n - k$ choices if k is odd. If k is even, we can either close the current cycle or choose a new number to go at the end of the current one. There are $n - k + 1$ choices in this case.

All together, if n is even, then the number of permutations of $[n]$ that contain only cycles of even length is

$$(n-1)^2(n-3)^2\cdots(1)^2 = \left(\frac{n!}{2^{n/2}(n/2)!}\right)^2 = \frac{n!}{2^n}\binom{n}{n/2}.$$

Thus, the probability of choosing a permutation uniformly at random and obtaining one that contains only cycles of even length when n is even is

$$\frac{1}{2^n}\binom{n}{n/2}. \qquad \blacksquare$$

Combinatorial proof of identity (2), given Claims 1 and 2. For any colored permutation σ_C, find the smallest element of $[n]$ contained in an odd-length cycle w of σ_C. Let $f(\sigma_C)$ be the colored permutation for which the color of w is flipped. Then $f(f(\sigma_C)) = \sigma_C$, and σ_C and $f(\sigma_C)$ have different parities for the number of red elements but the same probability of occurring. Thus, f is a sign-reversing involution on the colored permutations for which f is defined. The only colored permutations σ_C for which f is not defined are those that have only even-length cycles. By Claims 1 and 2,

$$\frac{1}{4^n}\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}(-1)^k = \begin{cases} \dfrac{1}{2^n}\dbinom{n}{n/2}, & n \text{ even;} \\ 0, & n \text{ odd,} \end{cases}$$

which is equivalent to identity (2).

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Solution to crossword puzzle with 96 clues

I	D	S	■	N	S	F	■	C	H	I	■	P	R	E
O	U	I	■	E	P	T	■	D	E	N	■	O	E	R
N	O	X	W	O	R	D	■	C	A	N	G	I	V	E
■	■	■	R	N	A	■	■	■	P	A	R	■	■	■
L	A	Z	Y	■	I	N	A	N	■	T	E	T	R	A
O	R	E	■	A	N	Y	M	O	R	E	■	A	O	L
B	E	N	C	H	■	E	Y	R	E	■	Q	U	I	P
■	■	■	A	A	A	■	■	■	L	O	U	■	■	■
A	J	A	R	■	I	O	W	A	■	N	O	W	A	Y
D	A	Y	■	C	L	U	E	S	T	O	■	A	L	E
O	M	E	G	A	■	T	E	P	E	■	S	N	I	T
■	■	■	E	S	S	■	■	■	T	W	O	■	■	■
A	I	D	T	H	A	N	■	T	H	I	S	O	N	E
R	A	I	■	I	R	A	■	K	E	N	■	L	E	A
M	M	D	■	N	I	P	■	O	R	K	■	D	O	T

Puzzle appears on page 535. Clues appear on page 536.