Honors 213

Fourth Hour Exam

Name ________________________________

Monday, April 24, 2000
100 points

1. Personalities (5 pts. each). Give brief descriptions of each of the following individuals, and (again, briefly) say what their contribution to the study of the parallel postulate is.

C.F. Gauss (1777-1855)

Nikolai Lobachevsky (1792-1856)

Janos Bolya (1802 – 1860)

Girolamo Saccheri (1667-1733)

2. (15 pts.) Do one (but only one) of the following two problems:

a. Show that in hyperbolic geometry if lines l and l' are parallel, and if A, B , and C are three distinct points on l, and further that if lines drawn from A, B, and C perpendicular to l' meet l' in points A', B', and C' respectively, then it can not happen that AA', BB', and CC' are all congruent to each other (Theorem 6.3).

b. Show (in neutral geometry) that in triangle ABC, the sums of the angle measures of angles A and B are less than 180 degrees (corollary 1 to theorem 4.3)

3. Equivalents to the parallel postulate

   a. (15 pts.) Show that Hilbert's Postulate is equivalent to the statement that if two lines are parallel and one is cut by a transversal, then so is the other (prop. 4.8)

   b. (5 pts.) From the point of view of a formal system of axioms, what does it mean to say that Hilbert's postulate and some other statement are equivalent?

4. Parallel lines (15 pts.) Give a definition of limiting parallel rays, and illustrate your definition in the Klein model.

5. (15 pts.) Give an explanation of our approach to the proof of the "metamathematical theorem", being careful to explain the role that Euclidean geometry plays in the proof.

6. (15 pts.) On page 226, our author writes that “Had Saccheri, Legendre, F. Bolyai, or any of the dozens of other scholars succeeded in proving Euclid V from the other axioms, with the noble intention of making Euclidean geometry more secure and elegant, they would have instead completely destroyed Euclidean geometry as a consistent body of thought.”

Give a careful explanation of this statement.

Axioms and major propositions

EP1: For every point P and for every point Q not equal to P there exists a unique line **l** that passes through P and Q

EP2: For every segment AB and for every segment CD there exists a unique point E such that B is between A and E and segment CD is congruent to segment BE.

EP3: For every point O and every point A not equal to O there exists a circle with center O and radius OA.

EP4: All right angles are congruent to each other.

Euclidean Parallel Postulate: For every line **l** and for every point P that does not lie on **l** there exists a unique line m through P that is parallel to **l**.

LR1: The following are the six types of justifications allowed for statements in proofs:

(1) By hypothesis (given)
(2) By axiom/postulate ...
(3) By theorem ... (previously proved)
(4) By definition ...
(5) By step ... (a previous step in the argument
(6) By rule ... of logic

LR2: To prove a statement H => C, assume the negation of statement C (RAA Hypothesis) and deduce an absurd statement using the hypothesis H if needed in your deduction.

LR3: The statement "not (not S)" means the same thing as "S".

LR4: The statement "not[H=>C]" means the same thing as "H and not C".

LR5: The statement "not[P and Q]" means the same thing as "not P or not Q".

LR6: The statement "not (forall(x) S(x))" means the same thing as "there exists(x) not(S(x))"

LR7: The statement "not (there exists(x) S(x))" means the same thing as "forall(x) not S(x)".

LR8: (modus ponens) If P => Q and P are steps in a proof, then Q is a justifiable step.

LR9: (a) []P => Q] & [Q => R]] => [P => R].
(b) [P & Q] => P, [P & Q] => Q.
(c) [~Q => ~p] <=> [p => Q]

LR10: For every statement P, "P or ~P" is a valid step in a proof.

LR11: Suppose the disjunction of statements S1 or S2 or ... or Sn is already a valid step in a proof. Suppose that proofs of C are carried out from each of the case assumptions S1, S2, ... Sn. Then C can be concluded as a valid step in the proof (proof by cases).

IA 1: For every point P and for every point Q not equal to P there exists a unique line l incident with P and Q.

IA 2: For every line l there exists at lease two points incident with l

IA 3: There exist three distinct points with the property that no line is incident with all three of them.

Prop 2.1: Non parallel distinct lines have a unique point in common

Prop 2.2: There exist three distinct non concurrent lines

Prop 2.3: For every line there is at least one point not lying on it.

Prop 2.4: For every point there is at least one line not passing through it.

Prop 2.5 : For every point P there exist at least two lines through P

BA 1: IF A*B*C then A, B, and C are three distinct colinear points, and C*B*A

BA 2: Given any two distinct points B and D, there exist points A, C, and E lying on the line through B and D such that A*B*D, B*C*D, and B*D*E.

BA 3: If A, B, and C are three distinct colinear points, then one and only one of the points is between the other two.

BA4: For every line l and for any three points A, B, and C not lying on l,

2. if A and B are on the same side of l and B and C are on the same side of , then A and C are on the same side of l

3. If A and B are on opposite sides of l and B and C are on opposite sides of l, then A and C are on the same side of l

(corr:) If A and B are on opposite sides of l and B and C are on the same side of l, then A and C are on opposite sides of l

Prop. 3.1: Lines the union of rays AB and BA, and the rays have only the segment AB in common.

Prop 3.2: Every line bounds exactly two half-planes and these half-planes have no point in common.

Prop 3.3: Given A*B*C and A*C*D then B*C*D AND A*B*D

corr: Given A*B*C and B*C*D then A*B*D and A*C*D.

Prop 3.4: IF C*A*B and l is the line through A, B, and C, then for every point P lying on l, P lies either on the ray AB or on the opposite ray AC.

Pasch's Theorem: If A, B, C are distinct noncollinear points and l is any line intersecting AB at a point between A and B, then l also intersects AC or BC. If C does not lie on l, then l does not intersect both AC and BC.

Prop. 3.6 Given A*B*C. Then B is the only point common to ray BA and ray BC, and ray AC = ray AC.

Prop 3.7: Given an an angle CAB and point d lying on line BC. Then D is in the interior of angle CAB if and only if B*D*C

Prop. 3.8: If D is in the interior of angle CAB then (a) so is every other point on ray AD except a; (b) no point on the opposite ray to AD is in the interior of angle CAB, and (c) if C*A*E then B is in the interior of angle DAE.

Crossbar Theorem: if ray Ad is between ray AC and ray AB, then ray AD intersects segment BC.

Prop. 3.9: (a) if ray r emanating from an exterior point of triangle ABC intersects side AB in a point between A and B, then r also intersects side AC or side BC. (b) If a ray emanates from an interior point of triangle ABC then it intersects one of the sides, and if it does not pass through a vertex it intersects only one side.

CA1: Copying segments onto rays

CA2: Congruency of segments is an equivalence relation

CA3: Addition of segments

CA4: Copying angles

CA5: Congruency of angles is an equivalence relation

CA6: SAS

corr: copying triangles

Prop 3.10: if in triangle ABC we have AB congruent to AC, then angle B is congruent to angle C.

Prop 3.11: Segment subtraction.

Prop 3.12: Given segment AC congruent to segment DF. Then for any point B between A and C, there is a unique point E between D and F such that AB is congruent to DE.

Prop 3.13: Segment ordering.

Prop. 3.14: Supplements of congruent angles are congruent.

Prop 3.15: Vertical angles are congruent to each other. An angle congruent to a right angle is a right angle.

Prop. 3.16: For every line l and for every point P there exists a line through P perpendicular to l

Prop 3.17: ASA congruence of triangles.

Prop 3.18: If in triangle ABC we have angle B congruent to angle C, then segment AB is congruent to segment AC, and triangle ABC is isosceles.

Prop 3.19: Angle addition.

Prop 3.20: Angle subtraction

Prop 3.21: Ordering of angles

Prop 3.22: SSS congruence of triangles.

Euclid's Proposition 1: Given any segment, there is an equilateral triangle having the given segment as one of its sides.

Circular Continuity Principle: if a circle C has one point inside and one point outside another circle C', then the two circles intersect in two points.

Elementary Continuity Principle: If one end point of a segment is inside a circle and the other outside, then the segment intersects the circle.

Archimedes' Axiom: IF CD is any segment, A any point, and r any ray with vertex A, then for every point B != A on r there is a number n such that when CD is laid off n times on r starting at A, a point E is reached such that nCD is congruent to AT and either B = E or A*B*E.

Aristotle's Axiom: Given any side of an acute angle and any segment AB there exists a point Y on the given side of the angle such that if X is at the foot of the perpendicular from Y to the other side of the angle, XY > AB.

Corr: Let ray AB be any ray, P any point not collinear with A and B, an angle XVY any acute angle. Then there exists a point R on ray AB such that angle PRA < angle XVY.

Dedekind's Axiom: Suppose the set {l} of points on a line l is the disjoint union of two nonempty subsets X and Y such that no point of one is between two points of the other. Then there exists a unique point O on l such that one of the subsets is equal to a ray of l with vertex O and the other set is equal to the complement.

Hilbert's Axiom of Parallelism: For every line l and every point P not lying on l there is at most one line m through P such that m is parallel to l.

(with many thanks for the student work involved in typing the following in)

Theorem 4.1 (alternate interior angle theorem): If two lines cut by a transversal have a pair of congruent alternate interior angles, then the two lines are parallel.

Corollary 1: Two lines perpendicular to the same line are parallel.

Corollary 2: if L is any line, and P is any point not on L, there exists at least one line M through P parallel to L.

Theorem 4.2 (exterior angle theorem): An exterior angle of a triangle is greater than either remote interior angle.

Proposition 4.1: SAA congruence

Proposition 4.2: Two right triangles are congruent if the hypotenuse and the leg of one are congruent respectively to the hypotenuse and the leg of the other.

Proposition 4.3 (midpoints): Every segment has a unique midpoint.

Proposition 4.4 (bisectors): A) every angle has a unique bisector. B) every segment has a unique perpendicular bisector.

Proposition 4.5: In a triangle ABC, the greater angle lies opposite the greater side and the greater side lies opposite the greater angle, i.e. AB greater than BC if and only if angle C is greater than angle A.

Proposition 4.6: Given triangle ABC and triangle A'B'C', if AB is congruent to A'B' and BC is congruent to B'C', then angle B is less than angle B' if and only if AC is less than A'C'.

Theorem 4.3:

2. There is a unique way of assigning a degree measure to each angle such that the following properties hold:
   1. the degree measure of angle A is a real number such that 0 < the degree measure of angle A < 180 degrees.
   2. 1. The degree measure of angle A = 90 degrees if and only if angle A is a right angle.
   3. 2. The degree measure of angle A = the degree measure of angle B if and only if angle A is congruent to angle B.
   4. 3. If ray AC is interior to angle DAB, then the degree measure of angle DAB equals the degree measure of DAC plus the degree measure of CAB.
   5. 4. For every real number X between 0 and 180 there exists an angle A s.t. the degree measure of angle A equals X degrees.
   6. 5. If angle B is supplementary to angle A, then the degree measure of angle A plus the degree measure of angle B equals 180 degrees.
   7. 6. The degree measure of angle A is greater than the degree measure of angle B if and only if angle A is greater than angle B.
3. Given a segment OI called a unit segment, then there is a unique way of assigning a length (AB) to each segment AB s.t. the following properties hold:
   8. (AB) is a positive real number and (OI) =1.

   8. (AB) = (CD) if and only if AB is congruent to (CD).

   9. A*B*C if and only if (AC) = (AB) + (BC).

   10. (AB) < (CD) if and only if AB < CD.

   11. For every positive real number X, there exists a segment AB s.t. (AB) = X.

Corollary 1: The sum of the degree measures of any two angles of a triangle is less than 180 degrees.

Corollary 2 (triangle inequality): if A, B, and C are three noncolinear points, then (AC) < (AB) + (BC).

Theorem 4.4 (Saccheri-Legendre): The sum of the degree measures of the three angles in any triangle is less than or equal to 180 degrees.

Corollary 1: The sum of the degree measures of any two angles in a triangle is less than or equal to the degree measure of their remote exterior angle.

Corollary 2: The sum of the degree measures of the angles in any convex quadrilateral is at most 360 degrees.

Euclid's Postulate V (Parallel Postulate): If two lines are intersected by a transversal in such a way that the sum of the degree measures of the two interior angles on one side of the transversal is less than 180 degrees, then the two lines meet on that side of the transversal.

Theorem 4.5: Euclid's fifth postulate if and only if Hilbert's parallel postulate.

Proposition 4.7: Hilbert's parallel postulate if and only if if a line intersects one of two parallel lines, then it also intersects the other.

Proposition 4.8: Hilbert's parallel postulate if and only if converse to theorem 4.1 (alt int angles)

Prop. 4.9: Hilbert's parallel postulate if and only if if T is a transversal to L and M, L is parallel to M, and T is perpendicular to L, then T is perpendicular to M.

Prop. 4.10: Hilbert's parallel postulate if and only if if K is parallel to L, M is perpendicular to K, and N is perpendicular to L, then either M = N, or M is parallel to N.

Prop 4.11: Hilbert's parallel postulate implies that the angle sum of every triangle is 180 degrees.

Theorem 4.6: Let triangle ABC be any triangle and D a point in between A and B. Then the defect of triangle ABC = the defect of triangle ACD + the defect of triangle BCD (additivity of the defect).

Corollary: Under the same hypothesis, the angle sum of triangle ABC is equal to 180 degrees if and only if the angle sums of both triangle ACD and triangle BCD are equal to 180 degrees.

Theorem 4.7: If a triangle exists whose angle sum is 180 degrees, then a rectangle exists. If a rectangle exists, then every triangle has angle sum = 180 degrees.

Corollary: If there exists a triangle with positive defect, then all triangles have positive defect.

Wallis's Postulate: Given any triangle ABC, and given any segment DE, there exists a triangle DEF having DE as one of its sides that is similar to triangle ABC. (Denoted triangle DEF ~ triangle ABC.)

Clairaut's Axiom: Rectangles exist.

Theorem 5.1: Hypothesis: for any acute angle A and any point D in the interior of angle A, there exists an angle through D and not through A that intersects both sides of angle A. Conclusion: the angle sum of every triangle is 180 degrees.

Axioms and theorems of Hyperbolic Geometry

Hyperbolic Axiom: In hyperbolic geometry there exists a line L and a point P not on L s.t. at least two distinct lines parallel to L pass through P.

Lemma 6.1: Rectangles do not exist.

Universal Hyperbolic Theorem: In hyperbolic geometry, for every line L and every point P not on L, there passes through P at least two distinct parallels to L.

Corollary: In hyperbolic geometry, for every line L and every point P not on L there are infinitely many parallels to L through P.

Theorem 6.1: In hyperbolic geometry, all triangles have angle sum less than 180 degrees.

Corollary: In hyperbolic geometry, all convex quadrilaterals have angle sum less than 360 degrees.

Theorem 6.2: In hyperbolic geometry, if two triangles are similar, they are congruent. (in other words, AAA is a valid criterion for congruence of triangles.)

Theorem 6.3: In hyperbolic geometry, if L and L' are any distinct parallel lines, then any set of points on L equidistant from L' has at most two points in it.

Theorem 6.4: In hyperbolic geometry, if L and L' are parallel lines for which there exists a pair of points A and B on L equidistant from L', then L and L' have a common perpendicular segment that is also the shortest segment between L and L'.

Lemma 6.2: The segment joining the midpoints of the base and summit of a Saccheri quadrilateral is perpendicular to both the base and the summit, and this segment is shorter than the sides.

Theorem 6.5: In hyperbolic geometry, if lines L and L' have a common perpendicular segment MM', then they are parallel, and MM' is unique. Moreover, if A and B are any points on L s.t. M is the midpoint of segment AB, then A and B are equidistant from L'.

Theorem 6.6: For every line L and every point P not on L, let Q be the foot of the perpendicular from P to L. Then there are two unique nonopposite rays PX and PX' on opposite sides of line PQ that do not meet L and have the property that a ray emanating from P meets L if and only if it is between ray PX and ray PX'. Moreover, these limiting rays are situated symmetrically about line PQ in the sense that angle XPQ is congruent to angle X'PQ.

Theorem 6.7: Given M parallel to L s.t. M does not contain a limiting parallel ray to L in either direction, then there exists a common perpendicular to M and L. (which is unique by theorem 6.5.)

Metamathematical Theorem 1: If Euclidean geometry is consistent, then so is hyperbolic geometry.

Corollary: if Euclidean geometry is consistent, then no proof or disproof of the parallel postulate from the rest of Hilbert's postulates will ever be found, i.e., the parallel postulate is independent of the other postulates.