Math 180 E

FINAL EXAM

NAME____

General Notes:

1. Show work.
2. Look over the test first, and then begin.
3. Calculators are not permitted on this exam.

Wednesday, Dec. 15, 2010
200 pts

I. Basics

The study of first-semester calculus is the study of real-valued functions of one real variable.

a. (5 pts.) What do we mean by a function? Give a definition.

One of the ways in which the study of calculus differs from the work we did in high school is the notion of a **limit**, which allows us to say a number of important things about functions.

b. (5 pts.) Give a $\delta - \varepsilon$ definition of $\lim_{x \to z} f(x) = L$

c. (10 pts.) Using the definition to prove that $\lim_{x \to 2}(2x+1) = 5$, find a $\delta > 0$ which works for $\varepsilon = \frac{1}{100}$ and show that it works.

d. (15 pts.) Limits can be useful in working out how functions behave at particular points and at infinity. Consider the function $y = \frac{x^2 - 1}{x^2 - 4}$

i. Identify x and y intercepts: That is, points at which the graph of the function crosses the x and y axis.

ii. Identify any vertical, horizontal, and oblique asymptotes

iii. Using this information, give a sketch of the graph of this function.

Once we have limits, we can talk about continuity, one of the most important characteristics of functions.

e. (5 pts.) Using limits, define carefully (mathematically) what it means for a function f(x) to be continuous at a point a.

f. (5 pts.) Continuous functions have a number of important properties. One of them is known as the Intermediate Value Theorem. State the Intermediate Value Theorem (with preconditions).

g. (5 pts.) We have spent most of the term talking about the derivatives of functions. Limits are once again useful in saying what a derivative is. Give a formal definition (as a limit) of the derivative of a function f(x).

h. (10 pts.) Now use the formal (limit) definition of a derivative to calculate the derivative of the function $f(x) = x^2 + 1$

i. (5 pts.) One of the most important properties of differentiable functions is known as the Mean Value Theorem for functions (not the Mean Value Theorem for Definite Integrals, which we will look at later). State the Mean Value Theorem with preconditions.

j (5 pts.) One of the interpretations of the Mean Value Theorem for differentiable functions says that a car traveling between points A and B must, at some time in the trip, be traveling at exactly the car's average velocity in the trip from A to B. Explain (briefly) how this works.

k (5 pts.) One of the important consequences of the Mean Value Theorem (for differentiable functions) is the statement that if two functions have the same derivative then they differ by only a constant. Why is this true?

II. Derivatives.

In the following, use the rules we have developed for derivatives of functions. That is, you do not need to use the definition of the derivative (we've already done that).

a. Derivatives of functions and basic rules for derivatives (5 pts. each). Find the derivative of each of the following functions:

i. $x^2 + 2x + 1$

ii. $e^x \sin(x)$

iii. $\dfrac{\cos(x)}{e^x}$

iv. $e^{\cos(x)}$

v. $\displaystyle\int_1^x \frac{dt}{1+t^2}$

b. With each differentiation formula we get (for free!) an antiderivative formula. Find the most general antiderivatives of each of the following functions (5 pts. each)

i. $\int (x^2 + 2x + 1)dx$

ii. $\int \cos(x)dx$

iii. $\int \frac{dx}{x}$

III. Applications of differentiation.

a. Application to curve sketching. (15 pts.) Consider the function $y = x^3 - 3x + 1$.

i. Find the critical points of this function.

ii. Using the second derivative test, classify the critical points as local maxima or minima, *saying why in each case.*

iii. Identify any points of inflection, again saying why they are points of inflection.

b. Related Rates (10 pts.) The radius of a spherical balloon is increasing at 10 inches/minute. How fast is the surface area of the balloon increasing when the radius is 5 inches? Recall that the surface area of a sphere is given by $S = 4\pi r^2$

c. Optimization (15 pts.) (15 pts.) (From Strauss, Bradley, and Smith <u>Calculus</u>)

Suppose that it costs us $C(x) = \frac{1}{8}x^2 + 4x + 200$ dollars to manufacture and distribute **x** units of some commodity, and that we can sell each one for a price of (49-x) dollars per unit for a total revenue R(x) = $x(49 - x)$ dollars for x units. Our profit is then P(x) = R(x) - C(x). For what value of x will we obtain the largest profit?

IV Some more integration.

a. Some definitions (5 pts. each, except for the last one (iv) which is 10 pts.)

i. A **partition** P of an interval [a, b]

ii. The **norm** of a partition P

iii. A **Riemann sum**

iv (10 pts.) Define (formal, mathematical, $\delta - \varepsilon$ definition) $\int_a^b f(x)dx$

b. (5 pts. each) Evaluate

i. $\int_0^1 (x^2 + x + 1)dx$

ii. $\int_0^{\pi} \sin(x)dx$

iii. $\int_1^e \frac{dx}{x}$

iv. Calculate the average value of the function sin(x) over the interval $[0, \pi]$