The Fundamental Group and Knot Groups

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Abstract

The field of topology, in a general sense, is the study of continuity, connectedness and boundaries and there has been an effort in the field to characterize topological spaces (a familar example of a topological space is $\mathbb{R}^2$). There is a bridge between topology and algebra which is an algebraic structure called the fundamental group and this group will help us characterize topological spaces. This paper will give a focused introduction to algebraic topology for mathematically mature undergraduates. We will begin by presenting some basics of topology and defining topological equivalence. Then we will construct the fundamental group of a topological space and introduce Seifert van Kampen's Theorem and mathematical knots in order to construct the fundamental group of the knot complement, better known as the knot group.

Basics of Topology

In order to understand the bridge between topology and algebra, we must first understand some basic ideas from topology.

Definition. A **topological space**, (X, U) is a set of points X with a collection of subsets U called open sets that satisfy the following axioms.

1. Any union of open sets is open
2. Any finite intersection of open sets is open
3. The entire set X and $\emptyset$ are open

Usually, we will refer to (X, U) as just X for conveinence.

Definition. We will consider two topological spaces X and Y to be equivalent when there exists a function $h : X \to Y$ that is one-one, onto, continuous and has a continuous inverse. This function h is called a **homeomorphism** and X and Y are called **homeomorphic** denoted $X \sim Y$.

Theorem. The relation of homeomorphism forms an equivalence relationon the set of topological spaces.
Proof. We need to show that the relation of homeomorphism, $\sim$, is reflexive, symmetric and transitive. Consider X, Y and Z to be topological spaces and the functions $f : X \to Y$ and $g : Y \to Z$ to be homeomorphisms.

1. Reflexivity: $X \sim X$ because the identity map, $i : X \to X$, is bijective and i and i^{-1} are continuous.

2. Symmetry: If $X \sim Y$ and $f : X \to Y$ is a homeomorphism then $f^{-1} : Y \to X$ is a bijective, continuous function and thus f^{-1} is a homeomorphism. Therefore $Y \sim X$.

3. Transitivity: If $X \sim Y$ and $Y \sim Z$ and $f : X \to Y$, $g : Y \to Z$ are homeomorphisms then $f \circ g : X \to Z$ is a homeomorphism.

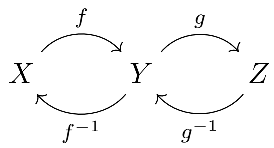

Figure 1

Therefore, the relation of homeomorphism is an equivalence relation.

□

Similar to how we have functions in real-space, we can have functions in a generic topological space. For constructing the fundamental group, we will introduce the functions paths and loops.

Definition. A **path** joining a to b in a topological space X is a continuous function $\gamma : [0,1] \to X$ where $\gamma(0) = a$ is called the beginning of the path and $\gamma(1) = b$ is the end.

We can obtain from this path, another path denoted γ^{-1} joining b to a defined by $\gamma^{-1}(t) = \gamma(1-t)$, for $t \in [0,1]$. This path γ^{-1} is simply the path γ in the reverse direction. When using paths, it is helpful to think of γ as describing the motion of a single point with the domain as time and the codomain as a position. A space is **path-connected** if any two points in the set can be joined by a path. A path that begins and ends at the same point, $\gamma(0) = \gamma(1) = a$ is called a **loop based at** a.

In particular, it is important to mention the **constant loop** (sometimes called the constant path) based at p denoted e and defined by $e(t) = p$ where $t \in [0,1]$.

Example. $\mathbb{R}^2$ is a topological space that is path-connected. A path in $\mathbb{R}^2$ might look something like this.

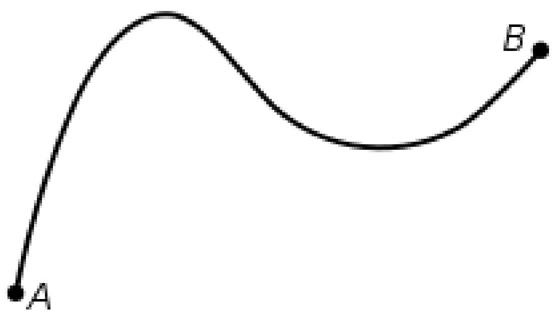

Figure 2

We can define the **product of two paths** as follows.

Definition. Consider α a path joining p to q and β a path joining q to r in a topological space X and $p, q, r \in X$. Then the path product is defined

$$\alpha \cdot \beta(t) = \begin{cases} \alpha(2t) & : 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & : \frac{1}{2} \leq t \leq 1. \end{cases}$$

When we take the product of two paths, we are essentially connecting one path to another at the ending of the first path and the begining of the second path. It is important to note that in order to have a product of two paths, we require that the ending of the first path be the begining of the second path. α ends at q and β begins at q so $\alpha \cdot \beta(t)$ is a continuous function that begins at p and ends at r. In the case for loops, if γ and δ are loops based at p then $\gamma \cdot \delta$ will be a loop based at p. Given the set of continuous functions that are loops based at p and a binary operation, it would be nice if these formed a group. Unfortunately, this binary operation isn't associative but we can resolve this problem after introducing homotopy.

Homotopy and the Fundamental Group

Definition. Let $f, g : X \to Y$ be continuous functions. Then f is **homotopic** to g if there exists a continuous function $F : [0,1] \times X \to Y$ such that $F(0,x) = f(x)$ and $F(1,x) = g(x)$ for all points $x \in X$. We write $f \underset{F}{\simeq} g$ and the function $F(t,x)$ is a **homotopy**. If $f(x)$ and $g(x)$ agree on a subset $A \subset X$ where A is nonempty (i.e. $f(a) = g(a)$ for $a \in A$), then there exists a homotopy F such that $F(t,a) = f(a) = g(a)$ for all $a \in A$ and $t \in [0,1]$. We then say that f **is homotopic to** g **relative to A** denoted $f \underset{F}{\simeq} g \ rel \ A$.

Example. In our situation, we will look at homotopic paths and loops in a particular topological space. If we have a homotopy $F : [0,1] \times [0,1] \to [0,1]$ such that $F(0,t) = \alpha(t)$ and $F(1,t) = \beta(t)$ where α and β are paths joining p to q then it is helpful to think of $F(s,t)$ as continuously deforming α into β with $s = 0$ being the start of the deformation and $s = 1$ being the end. The following figure illustrates this deformation.

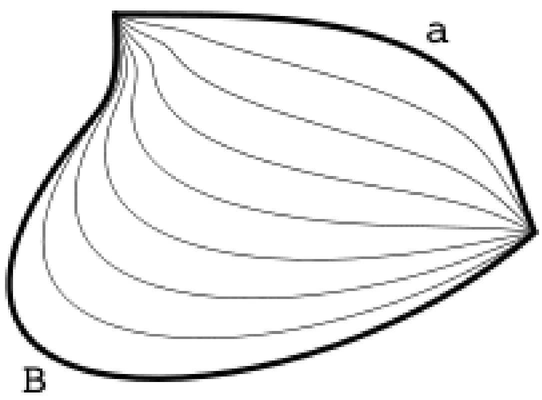

Figure 3

We can also imagine a similiar situation for loops.

Lemma. The relation of homotopy is an equivalence relation on the set of all maps from X to Y.

Proof. Consider continuous functions f, g and h from X to Y.

1. Reflexive: For any f we have $f \underset{F}{\simeq} f$ with $F(x,t) = f(x)$ for all $t \in [0,1]$.

2. Symmetric: If $f \underset{F}{\simeq} g$ then $g \underset{g}{\simeq} f$ where $G(x,t) = F(x, 1-t)$

3. Transitive: If $f \underset{F}{\simeq} g$ then $g \underset{g}{\simeq} h$ then $f \underset{H}{\simeq} h$ where

$$H(x,t) = \begin{cases} F(x,2t) & : 0 \le t \le \frac{1}{2} \\ G(x,2t-1) & : \frac{1}{2} \le t \le 1. \end{cases}$$

Therefore, the relation of homotopy is an equivalence relation.

□

The equivalence classes of the relation of homotopy are called the **homotopy classes** and are denoted $\langle\alpha\rangle$, the homotopy class of the loop α. We can use this new equivalence relation applied to loops to construct what is known as the fundamental group

Consider the set of all loops in a topological space X based at $p \in X$. By the previous lemma, the relation of homotopy relative to {0,1} is an equivalence relation on this set. Multiplication of loops induces multiplucation of the homotopy classes via

$$\langle\alpha\rangle \cdot \langle\beta\rangle = \langle\alpha \cdot \beta\rangle \tag{1}$$

Lemma. Loop multiplication of homotopy classes is well-defined.

Proof. Consider the loops α, α', β and β' in X such that $\alpha' \underset{F}{\simeq} \alpha$ rel $\{0,1\}$ and $\beta' \underset{F}{\simeq} \beta$ rel $\{0,1\}$, or $\langle\alpha'\rangle = \langle\alpha\rangle$ and $\langle\beta'\rangle = \langle\beta\rangle$. This means that there exists a homotopy $F : [0,1] \times [0,1] \to [0,1]$ such that $F(0,t) = \alpha'(t)$ and $F(1,t) = \alpha(t)$ and $F(s,a) = \alpha'(a) = \alpha(a)$ for $a \in \{0,1\}$. This is also true for β', β and $G(s,t)$, $(s,t \in [0,1])$. This implies that $\alpha' \cdot \beta' \underset{H}{\simeq} \alpha \cdot \beta$ rel $\{0,1\}$, where

$$H(s,t) = \begin{cases} F(2s,t) & : 0 \leq s \leq \frac{1}{2} \\ G(2s-1,t) & : \frac{1}{2} \leq s \leq 1. \end{cases}$$

H is a homotopy relative to $\{0,1\}$ so $\langle\alpha'\rangle \cdot \langle\beta'\rangle = \langle\alpha\rangle \cdot \langle\beta\rangle$ Therefore, loop multiplication is well-defined.

□

Under this multiplcation of homotopy classes, the set of homotopy classes of loops in X based at a point p forms a group.

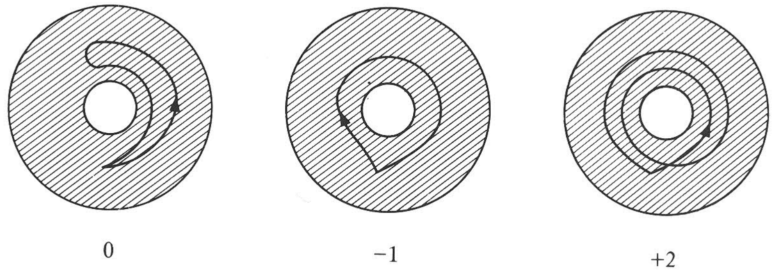

Figure 1.22

Figure 4

We should also consider what would happen the the fundamental group if we "filled" the hole of the annulus. Then there would be anything to "loop" around so the only element would be the trivial element.

Definition. The **fundamental group** of a set X based a p, denoted $\pi_1(X,p)$ is the set of homotopy classes of loops based at p with the binary operation of loop products.

Example. Consider the fundamental group of the annulus and refer to figure 4. The first element is in the homotopy class of the constant loop because under homotopy, we can imagine shrinking the loop down to a single point. The second loop is a nontrivial element that loops around the hole of the annulus once in a "negative direction." The third loop is a nontrivial element that loops around the hole of the annulus in a "positive direction." In a vague sense, the fundamental group describes how we "loop around stuff" in a topological space. The numbers below each loop relate the elements in the fundamental group of the annulus to elements of the group $\mathbb{Z}$. It turns out that these groups are isomorphic.

Theorem. The fundamental group is a group.

Proof. To show that the fundamental group is a group we will show associativity, identity element, inverse and closure since well-definedness has already been shown. Consider three loops α, β and γ in X based at p.

1. Closure:
   For closure, we only need to mention a few facts about loops that we already know. Consider loops α and β. Their product is

$$\alpha \cdot \beta(t) = \begin{cases} \alpha(2t) & : 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & : \frac{1}{2} \leq t \leq 1 \end{cases}$$

   We can see that $\alpha \cdot \beta$ is a continuous function that starts at p and ends at p so it is a loop based at p and therefore the homotopy class $\langle \alpha \cdot \beta \rangle$ is the homotopy class of a loop based at p.

2. Associativity:
   We must show that $(\langle \alpha \rangle \cdot \langle \beta \rangle) \cdot \langle \gamma \rangle = \langle \alpha \rangle \cdot (\langle \beta \rangle \cdot \langle \gamma \rangle) \implies \langle \alpha \cdot \beta \rangle \cdot \langle \gamma \rangle = \langle \alpha \rangle \cdot \langle \beta \cdot \gamma \rangle$. That is $(\alpha \cdot \beta) \cdot \gamma$ is homotopic to $\alpha \cdot (\beta \cdot \gamma)$. Consider a continuous function $F : [0,1] \times [0,1] \to X$ defined by

$$F(s,t) = \begin{cases} \alpha(\frac{4t}{1+s}) & : 0 \leq t \leq \frac{s+1}{4} \\ \beta(4t-s-1) & : \frac{s+1}{4} \leq t \leq \frac{s+2}{4} \\ \gamma(\frac{4t-s-2}{2-s}) & : \frac{s+2}{4} \leq t \leq 1 \end{cases}$$

   If we plug in $s = 0$ and $s = 1$ we get

$$F(0,t) = \begin{cases} \alpha(4t) & : 0 \leq t \leq \frac{1}{4} \\ \beta(4t-1) & : \frac{1}{4} \leq t \leq \frac{1}{2} \\ \gamma(2t-1) & : \frac{1}{2} \leq t \leq 1 \end{cases}$$

$$F(1,t) = \begin{cases} \alpha(2t) & : 0 \leq t \leq \frac{1}{2} \\ \beta(4t-2) & : \frac{1}{2} \leq t \leq \frac{3}{4} \\ \gamma(4t-3) & : \frac{3}{4} \leq t \leq 1 \end{cases}$$

   But, it we use the definition of loop products and write out $(\alpha \cdot \beta) \cdot \gamma$ and $\alpha \cdot (\beta \cdot \gamma)$ explicitly, we get

$$(\alpha \cdot \beta) \cdot \gamma = \begin{cases} \alpha(4t) & : 0 \leq t \leq \frac{1}{4} \\ \beta(4t-1) & : \frac{1}{4} \leq t \leq \frac{1}{2} \\ \gamma(2t-1) & : \frac{1}{2} \leq t \leq 1 \end{cases}$$

$$\alpha \cdot (\beta \cdot \gamma) = \begin{cases} \alpha(2t) & : 0 \leq t \leq \frac{1}{2} \\ \beta(4t-2) & : \frac{1}{2} \leq t \leq \frac{3}{4} \\ \gamma(4t-3) & : \frac{3}{4} \leq t \leq 1 \end{cases}$$

   So $(\alpha \cdot \beta) \cdot \gamma \underset{F}{\simeq} \alpha \cdot (\beta \cdot \gamma) \implies \langle \alpha \cdot \beta \rangle \cdot \langle \gamma \rangle = \langle \alpha \rangle \cdot \langle \beta \cdot \gamma \rangle$. Therefore loop products are associative.

3. Identity:
   Here, we claim that if $e(t)$ is based at p then $\langle e(t) \rangle$ is the identity element of the fundamental group. That is to say

$$\langle e \rangle \cdot \langle \alpha \rangle = \langle \alpha \rangle \cdot \langle e \rangle = \langle \alpha \rangle.$$

So we want to show that $e \cdot \alpha$ is homtopic to α. Consider the continuous function $G : [0,1] \times [0,1] \to X$ defined by

$$G(s,t) = \begin{cases} p & : 0 \leq t \leq \frac{1-s}{2} \\ \alpha(\frac{2t-1+s}{1+s}) & : \frac{1-s}{2} \leq t \leq 1 \end{cases}$$

If we consider $G(0,t)$ and $G(1,t)$ we get

$$G(1,t) = \begin{cases} p & : 0 \leq t \leq \frac{1}{2} \\ \alpha(2t-1) & : \frac{1}{2} \leq t \leq 1 \end{cases}$$

$$G(1,t) = \begin{cases} p & : 0 \leq t \leq 0 \\ \alpha(t) & : 0 \leq t \leq 1 \end{cases}$$

It is easy to see that $G(0,t) = e \cdot \alpha$ and $G(1,t) = \alpha$ therefore, $e \cdot \alpha \underset{G}{\simeq} \alpha$ and $\langle e \cdot \alpha \rangle = \langle \alpha \rangle$. By the definition of the loop product $\langle e \rangle \cdot \langle \alpha \rangle = \langle \alpha \rangle$. The other case for $\langle \alpha \rangle \cdot \langle e \rangle = \langle \alpha \rangle$ has a very similar arguement.

4. Inverses:
Recall the fact that for a loop γ, $\gamma^{-1}(t) = \gamma(1-t)$. We will show that $\langle \gamma \cdot \gamma^{-1} \rangle = \langle e \rangle$ or that $\gamma \cdot \gamma^{-1}$ is homotopic to e. Using the definition of the loop product we get

$$\gamma \cdot \gamma^{-1} = \begin{cases} \gamma(2t) & : 0 \leq t \leq \frac{1}{2} \\ \gamma(2t-2) & : \frac{1}{2} \leq t \leq 1 \end{cases}$$

Now consider the continuous function $H : [0,1] \times [0,1] \to X$ defined by

$$H(s,t) = \begin{cases} \gamma(2t(1-s)) & : 0 \leq t \leq \frac{1-s}{2} \\ \gamma((2-2t)(1-s)) & : \frac{1-s}{2} \leq t \leq 1 \end{cases}$$

If we compute $H(t,0)$ and $H(t,1)$ then we get

$$H(0,t) = \begin{cases} \gamma(2t) & : 0 \leq t \leq \frac{1}{2} \\ \gamma(2-2t) & : \frac{1}{2} \leq t \leq 1 \end{cases}$$

$$H(1,t) = \begin{cases} \gamma(0) & : 0 \leq t \leq 0 \\ \gamma(0) & : 0 \leq t \leq 1 \end{cases}$$

$H(0,t) = \gamma \cdot \gamma^{-1}(t)$ and $H(1,t) = \gamma(0) = e$. Therefore $\gamma \cdot \gamma^{-1} \underset{H}{\simeq} e$ and $\langle \gamma \cdot \gamma^{-1} \rangle = \langle e \rangle$. The other case for $\langle \gamma^{-1} \cdot \gamma \rangle = \langle e \rangle$ has a very similar arguement.

As we have defined it, the fundamental group of a topological space fulfills all the requirements of being a group.

□

Example of annulus. We can now consider the fundamental group of the annulus.

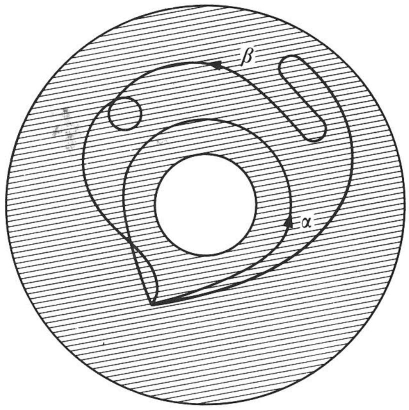

Figure 1.21

Figure 5

Now we have two big tools to help us characterize a topological space, homeomorphisms and the fundamental group. If we consider two topologocical spaces, X and Y, that are homeomorphic (topologically equivalent) wouldn't it be nice if their respective fundamental groups were related, say by an isomorphism? This happens to be true and we will prove it with the following theorem. We will not prove these theorems, so will have to trust M.A. Armstrong and take them as true.

Theorem. If X is path-connected then $\pi_1(X, p)$ is isomorphic to $\pi_1(X, q)$ for all $p, q \in X$.

This theorem allows us to conclude that the fundamental group does not depend on the base points of its elements if our space, X, is path-connected, so we can now write the fundamental group as $\pi_1(X)$, but we will still write $\pi_1(X, p)$ when we want to emphasize what the base point is.

So far, we have assigned each path-connected topological space to a group. But, we can also related these groups to eachother. Consider topological spaces X and Y. Let $f : X \to Y$ be continuous and p be the base point in X and $q = f(p)$ be the base point in Y of their respective fundamental groups. f induces a homomorphism

$$f_* : \pi_1(X, p) \to \pi_1(Y, q)$$

defined by $f_*(\langle \alpha \rangle) = \langle f \circ \alpha \rangle$.

With a little more work and exploiting this induced function, we can prove that *homeomorphic, path-connected spaces have isomorphic fundamental groups.*

Van Kampen's Theorem and Free Groups

We now have a general idea of what the fundamental group is and in some cases can reason through and figure out what the fundamental group explicitly is. But working with homotopy classes of loops and loop products is difficult and an easier way to express these groups would be nice. Unfortunately, the theorems of this section require more background in topology so we cannont fully prove them but we will present a variety of examples to show that the following theorms are plausible.

Definition. A **group with presentation** is a group expressed as $\langle S; R \rangle$ where S is a set of generators and R is a set of relations or relators. An element that is **relation** in R is an element that we can replace by the identity element, (or simply remove from a string of elements). If a pair of elements is a **relator** then one element of the pair can be replaced by the other. If $R = \emptyset$ then the group has no realtions.

Example The finite cyclic group of order 6 can be written as a group with presentation. $\langle a : a^6 \rangle$. In this case, a^6 is a relation and can be replaced by the identity. $aa^3a^6 = aa^3 = a^4$.

Example. Consider the space X that is an annulus and its fundamental group $\pi_1(X, p)$ (refer to figure 4). If a loop γ in X does not go around the hole in the middle of the annulus, then it is clear that under homotopy, we can deform γ into the single point p. Then the homotopy class of γ is the trivial element $\langle \gamma \rangle = \langle e \rangle$. Since the only nontrivial elements loop around the hole of the annulus, if α goes around the hole once in the positive direction, then it can generate the elements of the fundamental group. Its group presentation would be $\langle x : \emptyset \rangle$ where x represents the homotopy class of α.

Seifert-van Kampen's Theorem. Consider X, U_1 and U_2 path-connected topological spaces such that $U_1 \cup U_2 = X$ and $U_1 \cap U_2$ is nonempty and path-connected. If $p \in U_1 \cap U_2 \subset X$ and we know that

$$\begin{aligned} \pi_1(U_1 \cap U_2, p) &= \langle S; R \rangle \\ \pi_1(U_1, p) &= \langle S_1; R_1 \rangle \\ \pi_1(U_2, p) &= \langle S_2; R_2 \rangle \end{aligned}$$

then $\pi_1(X, p)$ is isomorphic to $\langle S_1 \cup S_2; R_1 \cup R_2 \cup R_S \rangle$ where $R_S = \{i_{1*}(s) = i_{2*}(s) : s \in S\}$, i_1 and i_2 are inclusion maps from $U_1 \cap U_2$ to their respective codomains and j_1 and j_2 are inclusion maps from their resepctive domains to X. Figures 6 and 7 illustrate this. Note: in some case, it will be valid to "disregard" the base point of these fundamental groups. We will exploit one of these cases in the construction of the knot group.

$$\begin{array}{ccc} U_1 \cap U_2 & \xrightarrow{i_1} & U_1 \\ \big\downarrow{\scriptstyle i_2} & & \big\downarrow{\scriptstyle j_1} \\ U_2 & \xrightarrow{j_2} & X \end{array} \quad \text{Figure 6}$$

$$\begin{array}{ccc} \pi_1(U_1 \cap U_2) & \xrightarrow{i_{1*}} & \pi_1(U_1) \\ \big\downarrow{\scriptstyle i_{2*}} & & \\ \pi_1(U_2) & & \end{array} \quad \text{Figure 7}$$

Example. Consider X to be a figure eight and U and V to be the peices of the figure eight as seen in the following figure and consider p the intersection of the two circles. We can figure out the fundamental group of U by the same reasoning as finding the fundamental group of the annulus. All nontrivial element in $\pi_1(U, p)$ is a loop that goes around the circle so $\pi_1(U, p) = \langle x : \emptyset \rangle$ where x represents a loop that goes around the circle in U once. Similarly $\pi_1(V, p) = \langle y : \emptyset \rangle$ where y represents a loop that goes around the circle in V once. By Van Kampen's theorem, $\pi_1(X) = \langle \{x, y\} : \emptyset \rangle$.

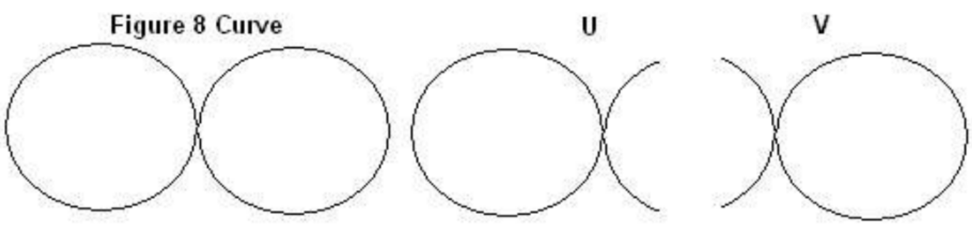

Figure 8

Knot Theory and Constructing the Knot Group

In order to know what the knot group is, we must know a little bit about what a mathematical knot is. A mathematical knot is like a normal knot that lies in space and is connected end to end but here's a more precise definition.

Definition. A **knot**, is a subspace of euclidean three-dimensional space which is homeomorphic to the cirlce. The simplest knot is called the *trivial knot* or the *unknot* and is just the unit circle in the (x, y) plane. In the following figure, we give a few examples.

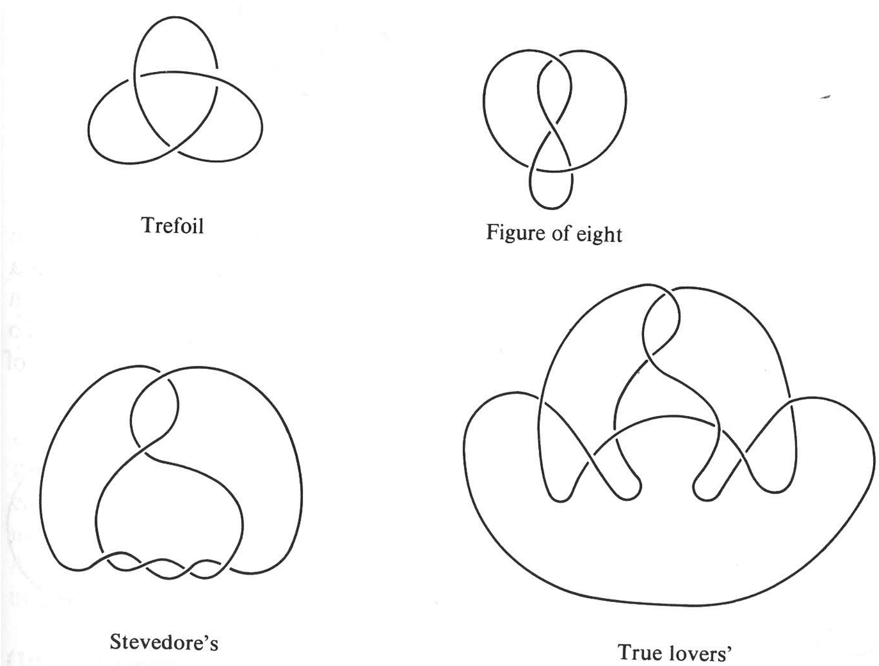

Figure 10.1

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Definition. Two knots k_1 and k_2 are considered equivalent if there exists a homeomorphism $h : \mathbb{E}^3 \to \mathbb{E}^3$ such that $h(k_1) = k_2$.

It is important to note, that in our definition, our homeomorphism is of the entire space that our knots k_1 and k_2 are embedded in. Knowing this, we can restrict h to $\mathbb{E}^3 - k_1$ to get a homeomorphism between $\mathbb{E}^3 - k_1$ and $\mathbb{E}^3 - k_2$. In essences, the two spaces around the knots are topologically equivalent. And if they are topologically equivalent, then they have the same fundamental group. This is where the knot group comes in.

Definition. Given a knot k, the **knot group** is the fundamental group of $\mathbb{E}^3 - k$.

Now, we begin the steps to actually constructing the knot group using the tools we have learned. First, under homeomorphism, we will deform our knot into an equivalent knot that has a nice projection and separates our "underpasses" and "overpasses". Then we will break up our space into pieces that have easy to compute fundamental groups and then construct the fundamental group of the entire space using Seifert Van Kampen's theorem. To help us visualize the process, we will use the trefoil knot as an example.

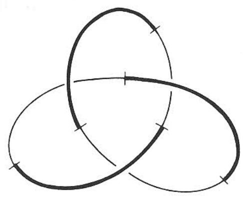

Figure 10

First, we will take a copy of our knot above the $z = 0$ plane in $\mathbb{E}^3$ that has a "nice" projection. Label the "overpasses" and "underpasses" of the knot, relative to the projection like in figure 10. The heavier lines represent "overpasses" while the lighter lines represent "underpasses". Now, take the underpasses and "stretch" them down to the $z = 0$ plane so that lie in the plane and are connected to the overpasses by perpendicular lines. Figure 11 will give a better visualization.

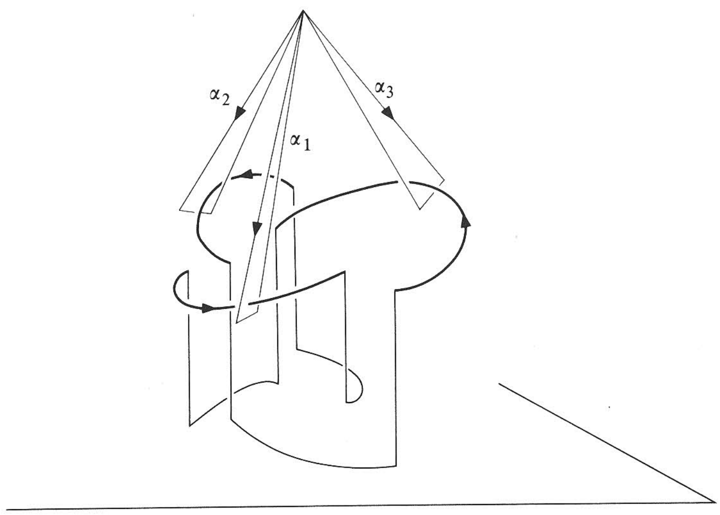

Figure 10.8

Figure 11

With this equivalent projection of our knot, we will take the first big piece of our space and calculate it's fundamental group. Take the closed half space $\mathbb{E}^3_+$ defined by $z \geq 0$ and give an orientation to k and pick a point p high above k in $\mathbb{E}^3_+$. We can now consider loops based at p and calcuate $\pi_1(\mathbb{E}^3_+ - k, p)$. For each overpass, consider a loop that is based at p and winds once around the overpass in a positive sense relative to the orientation of k (refer to figure 11). In this case, the positive sense can be determine with a "right-hand" rule. If you point the thumb of you right hand in the positive direction of the knot, then you fingers will curl in the positive sense for the loop the wind. Call these loops α_1, α_2, ... α_n for n overpasses. We will consider $\pi_(\mathbb{E}^3_+ - k, p)$ as a free group with generators $\{x_i\}$ where x_i represents the loops determined by α_i.

Lemma. $\pi_1(\mathbb{E}^3_+ - k, p)$ is the free group generated by x_1, x_2, ... x_n.

Proof. Let $\hat{k}$ denote the overpasses of k plus the vertical line segments which join their end points to the plane $z = 0$. Then clearly, $\mathbb{E}^3_+ - k$ and $\mathbb{E}^3_+ - \hat{k}$ have the same fundamental group. Then, for each overpass, we can make a vertical wall connecting the $z = 0$ plane to the overpass so that the wall is only "underneath" the overpass.

If we thicken these walls slightly in $\mathbb{E}^3_+$ then we get 3-d "balls" or subsets of $\mathbb{E}^3_+$ which we will call B_i for each of the n overpasses. We will do this in a way such that B_1, ... B_n are all disjoint.

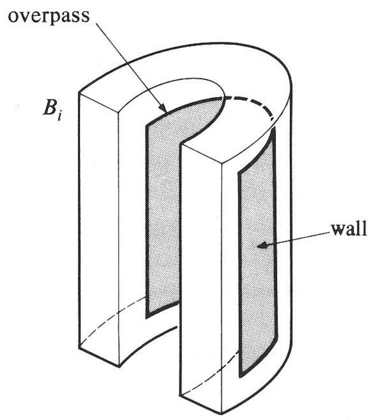

Figure 12

Now, suppose that we remove the interior of horseshoe-shaped base plus interior of each B_i from the closed half-space $\mathbb{E}^3_+$. The resulting space we will call X. We will build the space $\mathbb{E}^3_+ - \hat{k}$ as the union $X \cup (B_1 - \hat{k}) \cup ... \cup (B_n - \hat{k})$.

Let's take a look at the fundamental group of one of these pieces $(B_i - \hat{k})$. The basic shape of $B_i - \hat{k}$ is a short curved wall with a line (a piece of $\hat{k}$) taken out of the interior of it. If we think of loops in $(B_i - \hat{k})$ based at some point in $(B_i - \hat{k})$ then, similar to our example with the annulus, we can see that nontrivial loops will have to go around the line through B_i. We then get the free group $\langle\{x_i\};\emptyset\rangle$ as our $\pi_1(B_i - \hat{k})$.

If we consider our piece X, and consider loops based at some point in X, there is nothing for a loop to "hook around on" so every loop in X can be deformed under homotopy into the trivial loop. Therefore the only element in the free group for $\pi_1(X)$ is the identity element.

We can now apply Seifert Van Kampen's theorem to get $\pi_1(\mathbb{E}^3_+ - \hat{k})$. So

$$\pi_1(\mathbb{E}^3_+ - \hat{k}) \cong \langle\{x_1, x_2, ...x_n\};\emptyset\rangle.$$

It is important to note that even though the loops of the $\pi_1(B_i - k)$ and X didn't share a base point, we still applied Seifert Van Kampen's theorem. Unfortunately, it is hard to explain why we can still apply Seifert Van Kampen's theorem in this situation with the topology background presented but if the reader is interested, an explanation of why the base point is not important is given in Chapter 10 of M.A. Armstrong's *Basic Topology.*

□

We still need to add all of $\mathbb{E}^3_- - k$ (defined by $z \leq 0$) into our calculation of the knot group. We will now look at the underpasses of our knot. Consider the underpass between the ith and $(i+1)$th overpass and assume that kth overpass goes over our underpass as in figure 13. Similarly to how we created our B_i's, we thicken the projection of the underpass in $\mathbb{E}^3_-$ to create a 3-dimensional ball D_i. Consider adding the ball $D_i - k$ to $\mathbb{E}^3_+$ as in figure 13. To allow us to consider loops based at p in $D_i - k$, we add the lines from p to q and from q to r where r lies on the surface of D_i.

Let's consider the fundamental group of $D_i - k$ based at p using the two lines we constructed (refer to figure 12). The only property about $D_i - k$ that could allow us to create a nontrivial loop would be the line on the top of D_i where k used to be. But since k lies on the surface of D_i, there is no way we can loop around it so every loop in $D_i - k$ can be deformed under homotopy to the trivial loop. Therefore $\pi_1(D_i - k, p)$ is the trivial group. So by adding $D_i - k$ to $\mathbb{E}^3_+$, we added no new elements to the fundamental group.

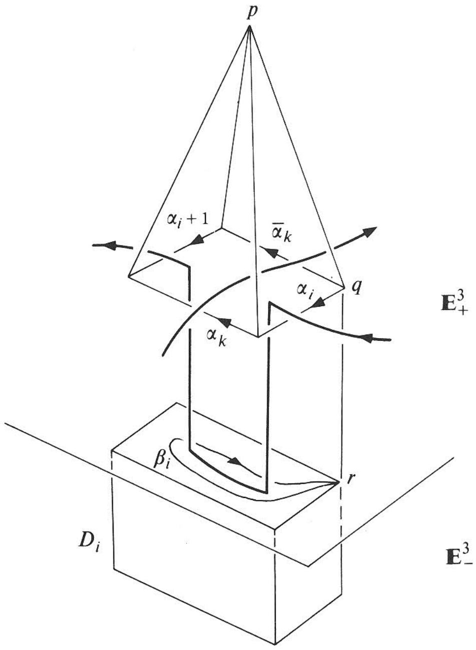

Figure 13

Looking at Van Kampen's theorem, we still need to consider the fundamental group of $D_i - k \cap \mathbb{E}^3_+$. This intersection is the rectangular surface on top of $D_i - k$ in the $z = 0$ plane. As before, we will add the lines from p to q and q to r so that we can base our points at p. We can create nontrivial loops in $D_i - k \cap \mathbb{E}^3_+$ based at p by winding around the missing segment of k, for example β_i in figure 13. The generator in $\pi_1(D_i - k \cap \mathbb{E}^3_+)$ corresponding to β_i will be denoted y_i. But we have to consider the relators in R_S. Consider the following maps from Van Kampen's theorem.

$$\begin{array}{ccc} \pi_1(\mathbb{E}^3_+ - k) \cap (D_i - k) & \xrightarrow{i_{1*}} & \pi_1(\mathbb{E}^3_+ - k) \\ \Big\downarrow{\scriptstyle i_{2*}} & & \\ \pi_1(D_i - k) & & \end{array}$$

Figure 14

$i_{2*}(\langle\beta_i\rangle) = \langle\beta_i\rangle = \langle e\rangle \in \langle\beta_i\rangle$ so $\langle\beta_i\rangle = \langle e\rangle \in \pi_1(\mathbb{E}^3_+ - k)$.

By simply sliding β_i vertically upwards, we obtain a loop homotopic to the product loop $\alpha_i\alpha_k\alpha_{i+1}^{-1}\bar{\alpha}_k^{-1} \simeq e$ in figure 14. If x_i represents the generator for α_i then we obtain the relation $x_i x_k x_{i+1}^{-1} x_k^{-1} = e$ or the relator $x_i x_k = x_k x_{i+1}$

The other possiblity is that our underpass is one which has been included two keep two overpasses apart (i.e. there is no overpass over this underpass). In this case β_i is homotopic to $\alpha_i\alpha_{i+1}^{-1}$. So we get another relation $x_i = x_{i+1}$.

If our knot has n underpasses then the first $n-1$ give us the relation for the fundamental group of

$$Y = (\mathbb{E}^3_+ - k) \cup (D_i - k) \cup ... \cup (D_{n-1} - k)$$

as $\langle\{x_1, x_2, ...x_n\} : R_1 \cup R_2 \cup ... \cup R_{n-1}\rangle$, where R_i is the relation we get at the ith underpass.

It turns out that this fundamental group is the fundamental group for the entire space, therefore $\pi_1(E^3 - k) = \langle\{x_1, x_2, \ ... \ x_n\} : R_1 \cup R_2 \cup ... \cup R_{n-1}\rangle$.

Example. The knot group of the trefoil knot. If we look at the over and underpasses of figure 10 then we have three generators x_1, x_2 and x_3 with the relations $x_1 x_2 = x_3 x_1$ and $x_2 x_3 = x_1 x_3$. If we write $a = x_1$ and $b = x_2$ then our group simplifies to $\langle \{a, b\} : aba = bab \rangle$.

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