Quaternion Algebras: History, Construction, and Application

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1 History

In October 1843, William Rowan Hamilton obsessed over a dilemma proposed by his son at breakfast, that of multiplying "triplets", or merely sets of three objects [1]. The better part of two centuries later, a method for such an operation may seem obvious. We have cross products and dot products for vectors, as well as fields of polynomials among other structures that involve multiplying objects of three or more parts. Now consider endowing the span (S) of the linearly independent set of three vectors over a field $\{\alpha, \beta, \gamma\}$ with an associative multiplication operation that distributes over addition. Take $\tau = (1)\alpha + (1)\beta + (1)\gamma$ as the simplest linear combination in S and compute $(\tau)^2$:

$$\begin{aligned}(\tau)(\tau) &= \alpha(\alpha+\beta+\gamma) + \beta(\alpha+\beta+\gamma) + \gamma(\alpha+\beta+\gamma)\\ &= (\alpha^2+\beta^2+\gamma^2) + \alpha\beta + \alpha\gamma + \beta\alpha + \beta\gamma + \gamma\alpha + \gamma\beta,\end{aligned}$$

a disastrous result if we haven't carefully defined ways to multiply elements of the spanning set. We need three symmetric bilinear products to absorb the last six terms of $(\tau)(\tau)$ into S. Hamilton was grappling with the hefty task of assigning these relations to such a vector space which would result in closure under multiplication and preserve other algebraic properties. The only similar structure that existed at the time was the complex numbers, and polynomial rings over multiple indeterminants, yet $\{\alpha, \beta, \gamma\}$ are meant to be determined objects, much like a coordinate system.

On one fateful walk Hamilton arrived at the following solution, which he then carved into the Brougham Bridge in Dublin [1]:

$$i^2 = j^2 = k^2 = ijk = -1 \tag{1}$$

which led to the following results:

$$ij = k;\ jk = i;\ ki = j;\ ji = -k;\ kj = -i;\ ik = -j. \tag{2}$$

From these Hamilton crafted the 4-dimensional Quaternion Algebra, $\mathbb{H}$, and proved that a 3-dimensional algebra of similar structure was impossible. He initially applied $\mathbb{H}$ to mechanics in three-dimensional space, after which a number of uses in physics were found for the ability of quaternions to induce rotations on physical objects under the influence of an additional attribute, such as time, force, temperature, or virtually any physical quality. Mathematically, the Quaternion Algebra over $\mathbb{R}$ is now know to have the largest dimension of any division algebra over $\mathbb{R}$, and the *only* such algebra up to isomorphism by Frobenius in 1878 [2]. Quaternion Algebras were generalized later to include the multiplicative relations above over an arbitrary field K of characteristic > 2, the construction of which is now possible in Sage with the option of setting $i^2 = a$ and $j^2 = b$ for with any $\{a, b\} \in K$ (an with $ij = k$).

With the Quaternions, Hamilton introduced the terms "vector" and "scalar" into the canon of mathematical terminology, as well as "associativity", when later a friend from college, John T. Graves, created the Octonions of dimension 8, where associativity fails [1]. In this paper I aim to "construct" the Quaternion Algebra over $\mathbb{R}$, discuss its geometry, provide insight into it's physical applications, and outline the proof of Frobenius' classification of division algebras over $\mathbb{R}$.

2 Preliminaries

An **Algebra** is a vector space over a field that is closed under an associative multiplication operation in addition to its properties as a vector space.

A **Quadratic Form** is a commutative map $((a, b) = \mu(b, a))$ $X \times X \to K$, where X is a finite-dimensional vector space over the field K. $N(a)$, taking the value (a, a) or $(a, a)^{1/2}$, is known as the *norm* of $a \in X$. A vector space is referred to as "normed" if it has a quadratic form. If $K = \mathbb{R}$, the norm gives X a partial ordering. [3]

If X is a **Division Algebra**, it is a vector space X over field K that can be treated as a division ring with no zero divisors if it is associative (the only cases we will consider). Thus X has multiplicative inverses, a multiplicative identity, is an abelian group under vector addition, and for $a, b \in X$, $ab = 0 \Rightarrow a = 0$ or $b = 0$. Multiplication in X is known as a **bilinear product**, $\beta : X \times X \to X$, meaning that for any $a \in X$, maps where $x \mapsto \beta(x, a)$ or $x \mapsto \beta(a, x)$ are linear for all $x \in X$. [3]

3 Quaternion Algebras

A **Quaternion Algebra** $\mathbb{H}$ is a normed (associative) division algebra of dimension 4 with quadratic form equivalent to the familiar inner product of vectors in $\mathbb{R}^n$, justifying our referring to $N(q)$ as the "magnitude" of q in $\mathbb{H}$. If $K = \mathbb{R}$, $N(q) = (q \cdot q)^{1/2}$. The bilinear map in $\mathbb{H}$ is the unique Quaternion Product computed below using results from (1).

3.1 Computations and Ring Structure

Elements in $\mathbb{H}$ can most readily be understood as linear combinations of basis elements $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k})$, written

$$q = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

for $(\mathbf{1}, a, b, c, d) \in K$, however it is often easier computationally to denote them as $q = [r, \mathbf{v}]$, where $r = \mathrm{re}(q) \in K$ is the scalar or "real" component of q, the coefficient of $\mathbf{1}$, and $\mathbf{v} = \mathrm{im}(q) = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ is known as the imaginary component. $q = [a, 0]$ implies that $q \in Re(\mathbb{H}) \cong \mathbb{R}$, while $q = [0, \mathbf{v}]$ implies $q \in Im(\mathbb{H}) \cong \mathbb{R}^3$. It is assumed that the reader is familiar with the operations of vector addition and scalar multiplication common to all vector spaces, as well as the cross and dot product of vectors in $\mathbb{R}^3$ from multivariate calculus, denoted $\mathbf{v}_1 \times \mathbf{v}_2$ and $(\mathbf{v}_1 \cdot \mathbf{v}_2)$, respectively. For the following operations, $\alpha \in K$ and $q_i \in \mathbb{H}$ with components alphabetized as the initial stating of q:

Inner Product and Norm

$$\begin{aligned}
(\mathrm{q}_1 \cdot q_2) &= a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2 \\
&= \mathrm{re}(q_1)\mathrm{re}(q_2) + (\mathrm{im}(q_1) \cdot \mathrm{im}(q_2)) \\
N(q_1) &= (q_1 \cdot q_1)^{1/2} \\
&= \sqrt{\mathrm{re}(q_1)^2 + N(\mathrm{im}(q_1))^2}
\end{aligned}$$

Quaternion Product

$$\begin{aligned}
q_1q_2 &= (a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k})(a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}) \\
&= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) \\
&+ (a_1b_2 + a_2b_1 - c_1d_2 - d_1c_2)\mathbf{i} \\
&+ (a_1c_2 - b_1d_2 + c_1a_2 - d_1b_2)\mathbf{j} \\
&+ (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)\mathbf{k} \\
&= [\mathrm{re}(q_1)\mathrm{re}(q_2) - \mathrm{im}(q_1) \cdot \mathrm{im}(q_1), \mathrm{re}(q_1)\mathrm{im}(q_2) + \mathrm{re}(q_2)\mathrm{im}(q_1) + \mathrm{im}(q_1) \times \mathrm{im}(q_2)]
\end{aligned}$$

To show that the $\mathbb{H}$ contains inverses, define the **conjugate** of $q \in \mathbb{H}$ to be $[\mathrm{re}(q), -\mathrm{im}(q)]$, and consider the following computation:

$$\begin{aligned}
q\left(\frac{\overline{q}}{N(q)^2}\right) &= [\mathrm{re}(q), \mathrm{im}(q)][\mathrm{re}(q), -\mathrm{im}(q)]\frac{1}{N(q)^2} \\
&= \frac{[(\mathrm{re}(q))^2 + \mathrm{im}(q) \cdot \mathrm{im}(q), \mathrm{re}(q)\mathrm{im}(q) - \mathrm{re}(q)\mathrm{im}(q) + \mathrm{im}(q) \times \mathrm{im}(q)]}{N(q)^2} \\
&= \frac{[N(q)^2, \mathbf{0}]}{N(q)^2} \\
&= 1
\end{aligned}$$

and we have the inverse of q defined to be $\frac{\overline{q}}{N(q)^2}$. Multiplication is associative (trust me) but not commutative, as shown by (1) and (2), while inner products are associative *and* commutative, which arises from basic field properties. An important quality of $\mathbb{H}$ borrowed from more familiar vector spaces is the homomorphic "norm property": the norm of a quaternion product is equal to the field product of the norms. Let $x_1 = [r_1, i_1]$, $x_2 = [r_2, i_2]$, and observe:

$$\begin{aligned}
N(xy) &= N([r_1r_2 - (i_1 \cdot i_2), r_1i_2 + r_2i_1 + (i_1 \times i_2)]) \\
&= ((r_1r_2)^2 - 2r_1r_2(i_1 \cdot i_2) + (i_1 \cdot i_2)^2 + (\{r_1i_2 + r_2i_1\} + i_1 \times i_2) \cdot (\{r_1i_2 + r_2i_1\} + i_1 \times i_2))^{1/2} \\
&= ((r_1r_2)^2 - 2r_1r_2(i_1 \cdot i_2) + (i_1 \cdot i_2)^2 + N(r_1i_2 + r_2i_1)^2 + 2(i_1 \times i_2 \cdot (r_1i_2 + r_2i_1)) + N(i_1 \times i_2)^2)^{1/2} \\
&= ((r_1r_2)^2 - 2r_1r_2(i_1 \cdot i_2) + (i_1 \cdot i_2)^2 + {r_1}^2N(i_2)^2 + {r_2}^2N(i_1)^2 + 2r_1r_2(i_1 \cdot i_2) + N(i_1 \times i_2)^2)^{1/2} \\
&= ((r_1r_2)^2 + {r_1}^2N(i_2)^2 + {r_2}^2N(i_1)^2 + N(i_1)^2N(i_2)^2)^{1/2} \\
&= ({r_1}^2 + N(i_1)^2)^{1/2}({r_2}^2 + N(i_2)^2)^{1/2} \\
&= N(x)N(y).
\end{aligned}$$

A bit messy, this vital property allows for many applications of the quaternions to geometry and physics. Notice that we are employing the the fact that $i_1 \times i_2 \perp \{i_1, i_2\}$, and the inner product of orthogonal vectors is zero, along with the following properties:

$$(i_1 \cdot i_2)^2 + N(i_1 \times i_2)^2 = N(i_1)^2N(i_2)^2 \tag{3}$$

$$i_1i_2 = -(i_1 \cdot i_2) + i_1 \times i_2 \tag{4}$$

3.2 Basic Substructures

$Re(\mathbb{H})$ is the set of all real quaternions. It contains the multiplicative identity $[1, 0]$, $1 \in K$, and for $x \in Re(\mathbb{H})$, x^{-1} and $-x$ are clearly contained in $Re(\mathbb{H})$, making it a subspace of $\mathbb{H}$. It is also the *center* of $\mathbb{H}$, making it a field. Let $c = [r_1, i_1]$ commute with all $q = [r_2, i_2] \in \mathbb{H}$. Then,

$$\begin{aligned}
0 = cq - qc &= [r_1r_2 - i_1 \cdot i_2, r_1(i_2) + r_2(i_1) + i_1 \times i_2] \\
&\quad - [r_2r_1 - i_2 \cdot i_1, r_2(i_1) + r_1(i_2) + i_2 \times i_1] \\
&= [0, i_1 \times i_2 - i_2 \times i_1] \\
&= [0, 2(i_1 \times i_2)] \Rightarrow c = q \text{ or } c = [r_1, 0]
\end{aligned}$$

This is a product of that fact that all pure components of quaternions can be visualized as vectors extending from the same origin, thus there exist no "parallel" quaternions to lead to a cross product of zero. We can form a trivial isomorphism $t : \mathbb{R} \to Re(\mathbb{H})$ defined by $t(r) = r[1, 0]$, and because of this $\mathbb{H}$ is a **central simple algebra**: its center is the field it is defined on, and it contains only trivial two-sided ideals (left to the orthogonal pure quaternions).

$Im(\mathbb{H})$ is then the set of all pure quaternions, equal to $Re(\mathbb{H})^{\perp}$, the **orthogonal complement** of $Re(\mathbb{H})$, meaning $\forall r \in Re(\mathbb{H})^{\perp}$ and $\forall i \in Re(\mathbb{H})$, $(r \cdot i) = 0$. $Im(\mathbb{H})$ is also a subspace, containing all additive inverses, and is isomorphic to any dimension 3 vector space over a field [6], yet for pure q_i, ${q_i}^2 = [c, 0]$ for some $c \in K$ and $q_1 q_2 = [-q_1 \cdot q_2, q_1 q_2]$, thus $Im(\mathbb{H})$ is not closed under multiplication unless we are considering only multiplication between orthogonal pure quaternions.

Lemma 1: $\mathbb{H} = Re(\mathbb{H}) \oplus Im(\mathbb{H})$.
Proof: As noted, $Re(\mathbb{H}) \cup Im(\mathbb{H}) = \{0\}$, and both $Re(\mathbb{H})$ and $Im(\mathbb{H})$ are subspaces. Together with the facts $Re(\mathbb{H})^{\perp} = Im(\mathbb{H})$, $Im(\mathbb{H})^{\perp} = Re(\mathbb{H})$ and $dim(\mathbb{H}) = 4 = 1+3 = dim(Re(\mathbb{H})) + dim(Im(\mathbb{H}))$, this is a valid direct sum decomposition of $\mathbb{H}$. So any element of $\mathbb{H}$ can be written as a sum of real and pure quaternions. As it turns out, in any finite dimensional division algebra A, such a decomposition is possible between the center of A and its orthogonal complement, the center being isomorphic to the field that A is defined on. Also for any division algebra, for imaginary q, $q^2 = [-c, 0]$ for some $c \in K$. Notice that all unit pure quaternions are solutions to the equation $x^2 = -1$. [3] [2].

$\mathbb{H}*$ is the set of nonzero quaternions, which is a non-abelian group under the quaternion product. $\mathbb{H}*$ contains the identity, inverses, and the quaternion product is associative.

$U(\mathbb{H})$ is the set of all **unit** quaternions ($N(q) = 1$), also known as **versors**. It is valid a subgroup of $H*$ (and a subspace of $\mathbb{H}$, but this is less exciting) because the norm property forces all versor products and inverses of versors to be versors, so $U(\mathbb{H})$ has inverses and is closed under the quaternion product. We will show how versors can act as geometric operators in the next section. [4]

4 Quaternion Geometry: $U(\mathbb{H}) \to SO(3)$

4.1 Geometric Preliminaries

We will denote the Euclidean n-dimensional space (E^n) by $\mathbb{R}^n$, because at most n coordinates are needed to locate each point in E^n, and it is useful to treat this region as a vector space over $\mathbb{R}$. A **similarity** on $\mathbb{R}^n$ is a map that preserves shape, such as the inner angles of a polygon, and the set of similarities on $\mathbb{R}^n$, a group under function composition, can be made the direct product of maps that preserve size (**isometries**) and maps that alter size. The group of isometries, $GO(n)$, contains all translations and reflections. Note that all rotations can be formed from a series of reflections [5]. Two objects are considered **congruent** if one can be transformed into another by a finite number of isometries. If L is a finite subgroup of congruences in Euclidean n-space, L can be described by a set of points that it fixes. [5]

The **General Orthogonal Group** ($GO(n)$) is the set of isometries of an n-dimensional Euclidean space with operation being the composition of mappings. [5]

The **Special Orthogonal Group** ($SO(n)$ or $Spin_n$) is the subgroup of GO(n) consisting of all **simple rotations**, or maps that fix $n-2$ dimensions. $SO(n)$ is *chiral*, meaning that it preserves orientation.

$\mathbf{mdot}_q$ is the general finite subgroup of $SO(n)$ containing all rotations about an axis q generated by $\theta = 2Pi/m$, $m \in \mathbb{Z}$.

4.2 Properties of Rotation

Rotations are bijections from $\mathbb{R}^n$ to $\mathbb{R}^n$, and so can be displayed as matrices [6]. By fixing $n-2$ dimensions, rotations are characterized by an angle θ about an axis-vector tangent to the plane of the 2 remaining dimensions. Therefore, if A is a rotation matrix corresponding to axis $\mathbf{v}$, then $A\mathbf{v} = \mathbf{v}$, and A has an 1 as an eigenvalue [6]. The inverse of a rotation can be denoted by either $-\theta$ or an antiparallel axis-vector.

Lemma 2: All rotations correspond to orthogonal matrices of determinant 1.
Proof. Recall that a matrix whose inverse is its adjoint (transpose, conjugate) is orthogonal. A rotation matrix A must preserve inner products, which can be written $\mathbf{v_1}^t\mathbf{v_2}$, thus we have

$$\begin{aligned} \mathbf{v_1}^t\mathbf{v_2} &= (A\mathbf{v_1}^t)A\mathbf{v_2} \\ &= \mathbf{v_1}^t(A^tA)\mathbf{v_2} \\ &\Rightarrow \mathbf{v_1}^t(I - A^tA)\mathbf{v_2} = 0 \\ &\Rightarrow A \text{ is Orthogonal,} \end{aligned}$$

and then,

$$\begin{aligned} (det(A))^2 &= det(A^t)det(A) \\ &= det(A^tA) = det(I) = 1, \end{aligned}$$

which implies $det(A) = 1$ because it is orientation preserving ($det(A) = -1$ would imply that A inverst the axis of rotation) [7].

4.3 Rotations by Group Action

Lets form the bijection $G : \mathbb{R} \to Im(\mathbb{H}_\mathbb{R})$, which is valid because vector spaces of equal dimension over the same field are isomorphic [6]. Now examine the map $\pi_{q_1,q_2} : \mathbf{v} \to q_1G(\mathbf{v})q_2$, for two othogonal, pure quaternions over of the reals. Let $q_1 = i_1$, $G(\mathbf{v}) = i_3$, and $q_2 = i_4$:

$$\begin{aligned}\pi_{q_1,q_2}(\mathbf{v}) = q_1 G(\mathbf{v}) q_2 &= (0, i_1)(-i_2 \cdot i_3, i_2 \times i_3) \\ &= (0, -(i_2 \cdot i_3)i_1 + i_1 \times (i_2 \times i_3)) \\ &= (0, -(i_2 \cdot i_3)i_1 + i_2(i_1 \cdot i_3) - i_3(i_1 \cdot i_2)) \\ &= (0, -(i_2 \cdot i_3)i_1 - (i_1 \cdot i_2)i_3)\end{aligned}$$

If we now apply G^{-1}, we have formed a similarity of Euclidean 3-space that multiplies magnitudes by $\sqrt{N(q_1)N(q_2)}$, remembering that $N(q_1 v q_2) = N(q_1)N(v)N(q_2)$ [5]. If $N(q_1 q_2) = 1$, we are dealing with unit, pure quaternions, and π becomes an isometry [5]. We have just gotten a taste of how geometric objects can be manipulated using quaternions, and we are leading up to the map in the title of this section.

Lemma 3: Any unit quaternion q can be written $q = [cos(\theta), sin(\theta)\mathbf{u}]$, for some $\theta \in \mathbb{R}$ and unit pure quaternion $\mathbf{u}$.
proof. We know that $q = [r, i]$, and that $(N(q))^2 = 1 = r^2 + i \cdot i$. Let $\theta = sin^{-1}(\sqrt{1-r^2})$, and $sin^2(\theta) = i \cdot i = 1 - r^2 \Rightarrow r = cos(\theta)$.
We can now simplify multiplication in $U(\mathbb{H})$ if we have two quaternions that share $\mathbf{u}$:

$$\begin{aligned}[cos(\theta), sin(\theta)\mathbf{u}][cos(\beta), sin(\beta)\mathbf{u}] &= (cos(\theta)cos(\beta) - sin(\theta)\mathbf{u} \cdot sin(\beta)\mathbf{u} \\ &+ (cos(\theta)(\mathbf{u}sin(\beta)) + cos(\beta)(\mathbf{u}sin(\theta) + \mathbf{u}sin(\theta) \times \mathbf{u}sin(\beta) \\ &= [cos(\theta+\beta), sin(\theta+\beta)\mathbf{u}],\end{aligned}$$

using Trigonometric identities. Now we can interject with the following observation:
For $m \neq 0 \in \mathbb{Z}$,

$$[cos(\frac{2\pi}{m}), sin(\frac{2\pi}{m})\mathbf{u}]$$

forms a finite subgroup of $U(\mathbb{H})$ that is isomorphic to $\mathbb{Z}_m$. (Proof left to reader).

Now finally, we can prove that $T : U(\mathbb{H}) \to SO(3)$ is a 2-1 homomorphism, which leads to the direct correspondence between the finite subgroups of $U(\mathbb{H})$ and the finite subgroups of $SO(3)$, $\{\mathbf{mdot}_q\}$:
Let $\alpha_u \in SO(3)$ denote a rotation of any $v \in \mathbb{R}^3$ by angle α about the unit vector $u \in \mathbb{R}^3$, acting as the axis of rotation. Because u and v form their own plane, Write $v = a + n$ for $a||u$ and $n \perp u$. Now using maps G and π_q above (where $q_1 = [cos(\frac{\alpha}{2}), sin(\frac{\alpha}{2})u]$, $q_2 = {q_1}^{-1} = \bar{q_1}$), we can show that $T(q) = \pi_q = \alpha_q$. Remember that multiplication in $\mathbb{H}$ distributes across addition, thus $\pi_q(v) = \pi_q(a) + \pi_q(n) = a + \pi_q(n)$, because a lies along

u, and so is unchanged by π_q. Now for $\pi_q(n)$ compute the following:

$$\begin{aligned}
\bar{q}(n)q &= (cos^2(\frac{\alpha}{2}) - sin^2(\frac{\alpha}{2})N(u)^2)n \\
&+ 2(u \cdot n)q + 2cos(\frac{\alpha}{2})sin(\frac{\alpha}{2})(u \times n) \\
&= (cos^2(\frac{\alpha}{2}) - sin^2(\frac{\alpha}{2}))n + 2cos(\frac{\alpha}{2})(u \times n) \\
&= cos(\alpha)n + sin(\alpha)n_\perp
\end{aligned}$$

where $n_\perp = (u \times n)$. It is not hard to tell that n has made a rotation of α about q, while maintaining its norm. The homomorphic property falls out of Lemma 3 and the well-defined composition of maps in $SO(3)$. The fact that $T(q) = T(-q)$ makes this a 2-to-1 surjective homomorphism. Therefore, any rotation in $SO(3)$ can be reduced to a single quaternion up to sign.

We can also call $\mathbb{R}^3 \cong Im(\mathbb{H})$ a $U(\mathbb{H})$-set by means of group action conjugacy of elements in $U(\mathbb{H})$, writing $(q, \mathbf{v}) \to q^{-1}\mathbf{v}q$. The two properties of group action in [8] are satisfied as follows:

1. $\forall \mathbf{v}, (1, \mathbf{v}) \to 1^{-1}\mathbf{v}1 = \mathbf{v}$
   $q_2(q_1, \mathbf{v}) \to (q_2, {q_1}^{-1}\mathbf{v}q_1) \to {q_2}^{-1}{q_1}^{-1}\mathbf{v}q_1q_2 = {q_2}^{-1}(\mathbf{w})q_2 = \mathbf{z}$

The "orbits" $(O(\mathbf{v}))$then have an especially nice interpretation: as spheres of radius $N(v)$, while the entire action of any $q \in U(\mathbb{H})$ is equal to a rotation of the entire three dimensional space about q by its corresponding angle.

4.4 Physical Applications

Hopefully it is now apparent that $\mathbb{R}$-valued quaternion algebras can be applied to physical and virtual scenarios. The simplest of these is the rotation of a rigid body in 3-space, and indeed Quaternions have been used in Aerospace technology to track satellite orbits and locate distant objects. They have also been used in computer graphics. A benefit of quaternions in real and virtual navigation is the avoidance of Gimbal Lock, a subject I will address in class. [7]

5 Frobenius' Theorem

The purpose of closing with this theorem is to put Quaternion Algebras, specifically those over the Reals, into a larger algebraic context. Hamilton initially opened up new mathematical pathways with his astounding display of possible extensions to the existing number fields, such that the following is often written:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \subset \mathbb{H}.$$

These pathways were soon after shown themselves to be finite, with each step further from $\mathbb{R}$ leading to the sacrifice of a vital algebraic attribute. From Real to Complex we lose ordering, from Complex to Quaternions we lose commutativity, and beyond the following limits established by Frobenius, the Octonions have been built at the sacrifice of associativity. We could continue, constructing then a 16-dimensional "algebra" that fails at something else, or we could work with what we have, which seems infinite nonetheless.

Theorem: Assume A is an algebra with unit and no zero divisors over $\mathbb{R}$. Then the following are equivalent:

1. The algebra A, considered as a vector space over $\mathbb{R}$, is finite dimensional.
2. Every element in A is algebraic.
3. Every element $x \in A$ can be uniquely represented as
$$x = r + z,\ r \in Re(A),\ z \in Im(A)$$
where $Im(A) = \{z \in A : z^2 \in \text{ and } z^2 \leq 0.\}$
4. Every element $x \in A$ is at most quadratic, such that it satisfies either
$$x - r = 0 \text{ or } (x-p)^2 + q^2 = 0$$
for $r, p, q \in \mathbb{R}$.
5. The algebra A is isomorphic to one of the following: $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

We will prove the theorem in a circular fashion, $n \Rightarrow n+1$, where (5) $\Rightarrow$ (1) immediately follows.

$1 \Rightarrow 2$ Let $dim(A) = n$. Then for $x \in A$,
$$\{1, x, x^2, x^3, ..., x^n\}$$
is a linearly depedent set, therefore $\exists\{\alpha_i\} \in \mathbb{R}$ such that
$$\sum_{i=0}^{n} \alpha_i x^i = 0.$$
Let $p(q) = \sum_{i=0}^{n} \alpha_i q^i$, and $p(x) = 0$.

$2 \Rightarrow 3$ As a division algebra, we know that $Z(A)$, the center of A, is isomorphic to the field it is defined on, namely $\mathbb{R}$, and that the direct sum of $Z(A)$ and $Z(A)^\perp$ gives us A. (Caution: if A is dimension 2, this direct sum is trivial because $Z(A) \cong \mathbb{C}$, but remember that elements in $\mathbb{C}$ are already of the form $z = Re(z) + Im(z)$. It is shown in [2] that this carries over to any vector space over $\mathbb{C}$.) Therefore any element $x \in A$ can take the form $x = r + i$ for $r \in Z(A)$ and $i \in Z(A)^\perp$. For uniqueness, assume $x = r' + i'$ as well, and that $r \neq r'$, $i \neq i'$. Then
$$i^2 = (r' - r + i')^2 = (r' - r)^2 + i'^2 + 2(r' - r)i'$$
and because $i^2 \in Z(A)$, $i' \in Z(A)$, a contradiction unless $r = r'$, which would imply that $i = i'$.

$3 \Rightarrow 4$ This step is a little more involved than it seems. We will need to extend some properties from $\mathbb{C}$ to higher dimensional division algebras. First, allow the fundamental theorem of algebra to hold for all division algebras [2]. A proof of this for $\mathbb{H}$ can be foung in [9] involving stereographic projections of limits at infinity on a 4-sphere, derivatives of quaternion-valued functions, and notions of analyticity in the 4-dimensional realm of $\mathbb{H}$. Define conjugation for A as in the complex world, $x = r - i$, and it is not hard to see that $\overline{q_1 + q_2} = \overline{q_1} + \overline{q_2}$ for $q_1, q_2 \in A$ by grouping real and imaginary components. The property $\overline{z_1 z_2} = (\overline{z_1})(\overline{z_2})$, for $z_i \in \mathbb{C}$ for higher dimensional algebras becomes $\overline{q_1 q_2} = \overline{q_2 q_1}$:

$$\begin{aligned}
\overline{q_1 q_2} &= [r_1 r_2 - i_1 i_2, -(r_1 i_2 + r_2 i_1 + i_1 i_2)] \\
&= [r_1 r_2 - i_1 i_2, -r_1 i_2 - r_2 i_1 + i_2 i_1)] \\
&= [r_2, -i_2][r_1, -i_1] \\
&= (\overline{q_2})(\overline{q_1}),
\end{aligned}$$

where it is understood that $-i_1 i_2 = i_2 i_1$ from vector calculus. Now we can prove the following lemma.

Lemma 4 If let $P(q) = a_0 + a_1 q + a_2 q^2 + \cdots + a_n q^n = 0$ for $q \in A$ and $a_i \in \mathbb{R}$, then $P(\overline{q}) = 0$.
Proof.

$$\begin{aligned}
0 = \overline{0} = \overline{P(q)} &= \sum_{i=0}^{n} \overline{a_i q^i} \\
&= \sum_{i=0}^{n} a_i \overline{q}^i
\end{aligned}$$

So roots from A come in conjugate pairs as in the Complex world.
Now lets examine the factorization of a generic polynomial $p(x)$ with real coefficients, indeterminant x and at most n roots from A. We know that $p(x)$ can be factored into irreducible polynomials with real coefficients, which are at most quadratic [8]. Knowing that roots from A come in conjugate pairs, for root $a \in A$ we have

$$\begin{aligned}
(x - a)(x - \overline{a}) &= x^2 - \overline{a}x - ax + a\overline{a} \\
&= x^2 - 2rx + r^2 - i \cdot i + ri - ri + i \times i \\
&= ((x - r)^2 - i \cdot i)
\end{aligned}$$

and $i \cdot i \in \mathbb{R}$, therefore quadratic factors take the desired form, with linear factors equal to $(x - m)$ for $m \in \mathbb{R}$.

$4 \Rightarrow 5$ First note that $Z(A)^{\perp}$ is a vector space over $\mathbb{R}$, if we consider that $\mathbf{0} \in A$ takes the form $[0,0]$. Now let us consider possible dimensions of I:
$Dim(I) = 0$: In this case $I = \{0\}$, and $A \cong \mathbb{R}$ by means of $r\mathbf{1} \to r$, for unit vector $\mathbf{1}$ in A.

$Dim(I) = 1$: Now take any $z \in I$, $z \neq 0$. Because $z = b\mathbf{e}_I$ for $b \in \mathbb{R}$ and basis element of I, $\mathbf{e}_I$, take any $a \in Z(A) \cong \mathbb{R}$, $az = ab(\mathbf{e}_I) \in I$. We know $z^{-1} \in Z(A)^{\perp}$ and so is also a multiple of $\mathbf{e}_I$, say $a\mathbf{e}_I$, thus using the commutativity of real numbers, $1 = zz^{-1} = \pm(ba)(\mathbf{e}_I)(\mathbf{e}_I) \Rightarrow {\mathbf{e}_I}^2 = \pm 1$ and because $\mathbf{e}_I^2 = +1$ would imply $\mathbf{e}_I = \pm 1$, we know ${\mathbf{e}_I}^2 = -1$. Thus if $Dim(I) = 1$, and $A \cong \mathbb{C}$.
$Dim(I) \geq 2$: In this case A has at least three linearly independent vectors $\{1, e_1, e_2\}$ that make up a basis for I. From the existence of inverses and our previous cases, we know that ${e_1}^2$ and ${e_2}^2$ exist in $Z(A)$ as $-\mathbf{1}$. Now consider a relation of linear dependence

$$\alpha e_1 + \beta e_2 + \gamma e_1 e_2 = 0.$$

A nontrivial solution $\{\alpha, \beta, \gamma\} \in \mathbb{R}$ would first imply that $\gamma \neq 0$, because we already know that $\alpha e_1 + \beta e_2 = 0$ has only trivial solutions. So we have,

$$-e_2 = \frac{\alpha}{\gamma} + \frac{\beta}{\gamma e_1} e_2,$$

but because any element of A can be separated into Real and Imaginary parts, and $-e_2 \in I$, we have $\alpha = 0 \Rightarrow -1 = \frac{\beta}{\gamma} {e_1}^{-1} \Rightarrow e_1 \in \mathbb{R}$, a contradiction. So if $Dim(I) \geq 2$, $Dim(I)$ is at least 3, such that $\{1, e_1, e_2, e_1 e_2\}$ make a linearly independent set. We now have a basis satisfying Hamilton's original quaternion relations, therefore any 4-dimensional division algebra is isomorphic to $\mathbb{H}$. If we suppose that a greater-dimensional division algebra exists, it can be shown that no multiplication rules exist that lead to a closed algebra that maintains associativity [2].

$5 \Rightarrow 1$ This last step is implied, and we have completed the proof.

6 Conclusion

Quaternion Algebras are a versatile tool for both purely algebraic study and physical application. They represent an area of linear algebra that has been completely demystified, but their failure to provide associativity has made them cumbersome objects to incorporate into the mainstream of tools for modeling physical phenomenon. Considering that a 9-part rotation matrix can be reduced to a 4-part quaternion, however, $\mathbb{H}$ over the Reals is worth being familiar with if a rigorous classification of spacial manipulations is desired.

7 Exercises

1. Prove that for any pure quaternion $q \in \mathbb{H}_{\mathbb{R}}$, the map $\rho_q : Im(\mathbb{H}_{\mathbb{R}}) \to Im(\mathbb{H}_{\mathbb{R}})$ given by $\rho(x) = -qxq^{-1}$ is a bijection, and is equal to a reflection in $GO(3)$ (remember that reflections correspond to a fixed plane).
2. Prove that the following matrix serves as an automorphism of $\mathbb{H}$ corresponding to a single quaternion. What is that quaternion?
$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

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