

Symmetry and Islamic Art

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Background

Islamic art is unique because its form and function permeate both distance and time. Since the birth of Islam, Muslim artists around the world have created pieces of art that share an uncanny resemblance to pieces created by other Muslim artists through common themes, appearances, and applications. These similarities include calligraphic elements, exaggerated depictions of plants and animals, and repetitive, often star-shaped, geometric patterns.

Seyyed Hossein Nasr, author of *Islamic Art and Spirituality*, proposes that the similarities between different pieces of Islamic art across time arise because of the connections between Islamic worship and art; the contemplation of Allah recommended in the Qur'an and the contemplative nature of Islamic art; and, finally, the remembrances of Allah as the final goal of Islamic worship and the role played by art in the lives of Muslim individuals and communities. He continues on to say that Islamic revelation is "crystalized in" art, which, in turn, guides the viewer toward Islamic revelation, so it is natural that Islamic art would take on a form that allows the artist and viewer to feel connected to Allah.



Figure 1: A few examples of applied Islamic geometrical patterns.

In particular, Islam emphasizes the oneness of Allah as the only god (called Unity), which is embodied in the form of geometric Islamic art. The Muslim artist attempts to express "the manifestation of Unity upon the plane of multiplicity" in his artwork, reflecting their beliefs in Unity, the dependence of all beings on Allah, the fleetingness of the physical world, and the "positive qualities of cosmic existence."

The popularity of geometric patterns to express Islamic revelation stems from a number of sources. First and foremost, the Qur'an bans the use of many images, including the representation of both Allah and the prophet Mohammed. Titus Burckhardt writes in *Sacred*

Art in East and West that “a Muslim’s awareness of the Divine presence is based on a feeling of limitlessness; he rejects all objectification of the Divine, except that which presents itself to him in the form of limitless space,” so the same geometric shape will be seen repeating over and over again on any given space.

In essence, Islamic art seeks to maintain truth in its expression and to not add to or subtract from the value of what is being represented. Repetitive geometric patterns are traditionally constructed by Muslim artists using only a ruler and compass. Thus, geometric patterns show no more and no less than their value; the possibilities for symmetries are bounded by the nature of the two-dimensional surface on which they are drawn. Burckhardt goes so far as to say that geometrical figures contained in a circle are the best possible visual symbol of the movement from Unity to multiplicity, although he does not back up this claim.

Due to their heavy use and reverence of geometric patterns, Muslim artists and mathematicians paved the way for much of the algebra we use today. From their reliance on a ruler and compass to create shapes on a two-dimensional surface, Muslim artists eventually created all possible symmetries that can be produced in that manner.

Disregarding the apparent visual differences between geometric patterns, there are only seventeen possible symmetries of repeating geometric figures on a two-dimensional plane. Of these seventeen symmetry groups, five are commonly found in Islamic art: p6m, p4m, cmm, pmm, and p6, with p6m and p4m appearing most often. We will discuss what the symbols mean later. The remaining twelve are not used nearly as frequently, and sometimes not at all, by Muslim artists, so we will exclude them from our analysis.

We will begin by exploring the method used to create Islamic geometrical patterns, and from there will look at the symmetry groups that arise from their structures.

Creating Islamic Patterns

As previously stated, Muslim artists in different periods of time and locations produced geometric patterns with only the help of a ruler and compass. With these two tools they were able to create many patterns, but the first step in all cases is to draw a circle. From there, polygons can be formed with careful division of the circle’s circumference and the addition of straight lines connecting the appropriate points. The polygons are then manipulated to create a variety of images. These images become patterns when they are repeated across a surface by connecting certain points of intersection with straight lines. The most common types of geometric patterns utilized in Islamic art are created by dividing the circle into pieces that number four or six, and multiples thereof.

We can divide a circle into four (or multiples of four) equal parts by first constructing two perpendicular diameters and using the endpoints of the diameters as the center for four intersecting semi-circles that share the radius of the circle. Next, connect opposite semi-circle intersection points by a diagonal through the center of the circle, as well as the edges of the circumscribing square. From here, there are two ways to create octagonal squares within the circle. One method, seen on the left-hand side of figure 2, is to connect every second intersection point that was created through the above process. The other involves connecting every third intersection point, and is exemplified on the right-hand side of figure 2.

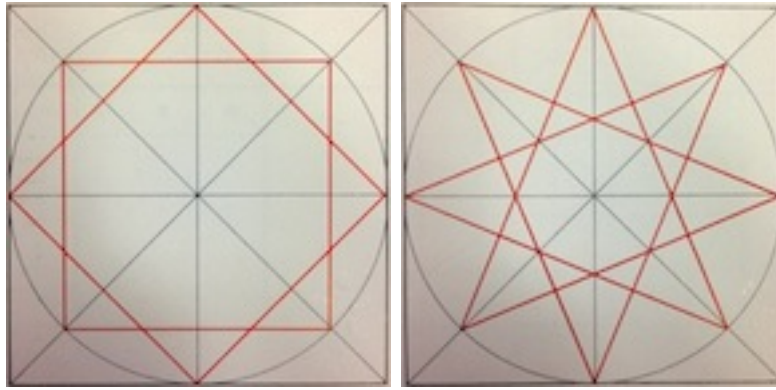


Figure 2: The two most commonly used bases for patterns with $4n$ parts. Left: Every second intersection point joined. Right: Every third point joined. Source: [El-Said, 1993]

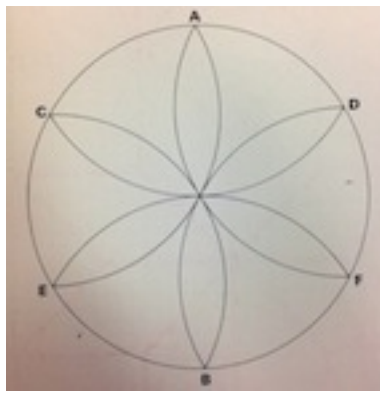


Figure 3: The petal shape created by intersecting a circle with 6 equally spaced arcs. Source: [El-Said, 1993]

The division of a circle into three (or multiples of three) equal parts begins similarly with the construction of two perpendicular diameters. This time, however, we use only the endpoints of one diameter as the center for two arcs that, again, maintain the radius of the circle. Beginning by adopting the four new points of intersection created by the arcs as respective centers, draw four more arcs, creating a petal shape (figure 3).

To complete the formulation of the two most commonly used formations, create two equilateral triangles within the circle by connecting the appropriate “petal tips.” The alternative version is made by constructing three new diameters through the intersection points of the triangles. The points where these diameters meet are then used as the vertices for two equilateral triangles. This is more clearly explained by figure 4.

These four bases can be manipulated by further division, drawing, and erasing to create the vast number of geometric, symmetrical patterns found in Islamic art. We will proceed by seeing how these patterns guide us to analysis of symmetries using group theory.

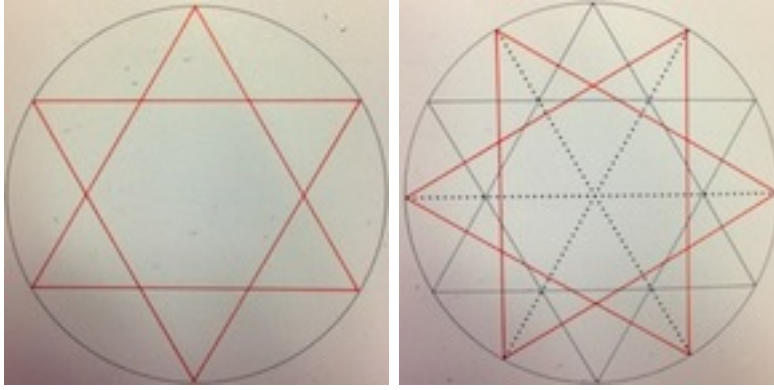


Figure 4: The two most commonly uses bases for patterns with $3n$ parts. Left: Initial equilateral triangle placement. Right: Alternative structure derived from (left). Source: [El-said, 1993]

Symmetries in E_2

The definition of **symmetric** we will be using is as follows: an object is symmetric if the position it occupies in space remains invariant to one or more isometry transformations, meaning that the appearance of the object does not change when modified by a translation, rotation, reflection, or glide reflection.

This leads naturally to the concept of a **symmetry**, which is a transformation that leaves the appearance of a particular object unchanged. The **symmetry group** (not to be confused with the symmetric group) of an object refers to the combination of isometry transformations that are symmetries of that object. **Wallpaper patterns** are simply repeating patterns in E_2 , the symmetry group of functions that preserve distance in two-dimensional Euclidean space. **Wallpaper groups** are symmetry groups which send a given pattern to itself, classified by a subgroup of E_2 . In order to analyze wallpaper groups, we must first discover the properties of E_2 .

A Euclidean transformation of \mathbb{R}^2 is a function of the form $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$T(\mathbf{x}) = \mathbf{a} + M\mathbf{x},$$

where $\mathbf{a}, \mathbf{x} \in \mathbb{R}^2$ and $M \in O_2$. The vector \mathbf{x} is the vector on which the transformations are performed. M and \mathbf{a} determine how the vector will be translated, rotated, or reflected. Again, isometries can be broken up into translations, rotations, and reflections.

Translations

The symmetry group of a given pattern always contains translational symmetries which describe the directions in which we can move the pattern, so it lands on top of itself. There are an infinite amount of directions in which the pattern can slide, but there must be two translations with the shortest magnitudes, \mathbf{a} and \mathbf{b} . When combined together, these translations can create any other possible translation, so they are a basis for all translations.

Given a vector \mathbf{a} , we define a function $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$T(\mathbf{x}) = \mathbf{a} + \mathbf{x}.$$

T translates \mathbf{x} by \mathbf{a} , which is the form of any translation. The set of all translated points forms a lattice spanned by \mathbf{a} and \mathbf{b} [Amstrong: 1988]

Rotations

Rotational symmetries are formed by rotating the pattern around a given point until it returns to itself. They often occur at centers of polygons and star shapes, as well as vertices and midpoints of sides. It turns out that an infinitely repeated pattern can only have 2-fold, 3-fold, 4-fold, and 6-fold rotational symmetries; this is due to a restriction imposed by group called the **crystallographic restriction**, which is proven later.

Let $A_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. Given a matrix A_θ , and an angle θ , we define a rotation $R : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$R(\mathbf{x}) = A_\theta \mathbf{x}.$$

$A_\theta \mathbf{x}$ is the counterclockwise rotation of \mathbf{x} by θ .

Mirror Reflections

Mirror reflections arise when a pattern can be reflected over a line and remain unchanged.

Let $B_\gamma = \begin{bmatrix} \cos(\gamma) & \sin(\gamma) \\ \sin(\gamma) & -\cos(\gamma) \end{bmatrix}$. Given a matrix B_γ , and an angle γ , we define a reflection $M : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$M(\mathbf{x}) = B_\gamma \mathbf{x}.$$

In this scenario, B_γ takes \mathbf{x} and reflects it over a line that is rotated $\frac{\gamma}{2}$ radians from the horizontal axis.

Wallpaper Groups

Isometries of wallpaper patterns are denoted by an ordered pair (\mathbf{a}, M) with $\mathbf{a} \in \mathbb{R}$ and $M \in O_2$, the group of orthogonal 2×2 matrices, where if $g = (\mathbf{a}, M)$, then

$$g(\mathbf{x}) = \mathbf{a} + M\mathbf{x} \quad \forall \mathbf{x} \in \mathbb{R}^2.$$

Note that the product of two such isometries is

$$(\mathbf{a}_1, M_1)(\mathbf{a}_2, M_2) = (\mathbf{a}_1 + M_1\mathbf{a}_2, M_1M_2).$$

We can then define a function $t : E_2 \rightarrow O_2$ by $t(\mathbf{a}, M) = M$, which is a homomorphism because

$$\begin{aligned}
t((\mathbf{a}_1, M_1)(\mathbf{a}_2, M_2)) &= t(\mathbf{a}_1 + M_1\mathbf{a}_2, M_1M_2) \\
&= M_1M_2 \\
&= t(\mathbf{a}_1, M_1)t(\mathbf{a}_2, M_2)
\end{aligned}$$

The kernel of t consists of the isometries $(\mathbf{a}, I) \forall \mathbf{a} \in \mathbb{R}^2$, the same form of the translations. Given a group G a subgroup of E_2 , we have two definitions. The **translation subgroup** of G is $H = G \cap T$, where T is the subgroup of E_2 that consists of translations. The **point group** of G is $J = t(G)$.

This brings us to the formal definition of a wallpaper group. A subgroup of E_2 is a **wallpaper group** if its translation subgroup is generated by two independent translations and its point group is finite.

The Crystallographic Restriction

We now have the tools to prove the crystallographic restriction. The definition of the crystallographic restriction given above can be expressed using abstract algebra terminology to say that the order of a rotation in a wallpaper group can only be 2, 3, 4, or 6. We adapt the proof from Armstrong [Armstrong, 1988].

Before launching into the proof, we first consider a few examples. Let q be the order of a rotation through $\frac{2\pi}{q}$ and $A_{\frac{2\pi}{q}} = \begin{bmatrix} \cos(\frac{2\pi}{q}) & -\sin(\frac{2\pi}{q}) \\ \sin(\frac{2\pi}{q}) & \cos(\frac{2\pi}{q}) \end{bmatrix}$ be the matrix associated with this rotation.

Example 1: $q = 3$

Let $q = 3$. First, we take an arbitrary vector $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}_2$ and multiply it by $A_{\frac{2\pi}{3}}$ to rotate it about the origin by $\frac{2\pi}{3}$ radians.

$$\begin{bmatrix} \cos(\frac{2\pi}{3}) & -\sin(\frac{2\pi}{3}) \\ \sin(\frac{2\pi}{3}) & \cos(\frac{2\pi}{3}) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos(\frac{2\pi}{3}) - y\sin(\frac{2\pi}{3}) \\ x\sin(\frac{2\pi}{3}) + y\cos(\frac{2\pi}{3}) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x - \frac{\sqrt{3}}{2}y \\ \frac{\sqrt{3}}{2}x - \frac{1}{2}y \end{bmatrix}$$

Now, rotate the new vector by another $\frac{2\pi}{3}$ radians.

$$\begin{bmatrix} \cos(\frac{2\pi}{3}) & -\sin(\frac{2\pi}{3}) \\ \sin(\frac{2\pi}{3}) & \cos(\frac{2\pi}{3}) \end{bmatrix} \begin{bmatrix} -\frac{1}{2}x - \frac{\sqrt{3}}{2}y \\ \frac{\sqrt{3}}{2}x - \frac{1}{2}y \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x + \frac{2\sqrt{3}}{4}y \\ -\frac{2\sqrt{3}}{4}x - \frac{1}{2}y \end{bmatrix}$$

Rotate the vector one more time.

$$\begin{bmatrix} \cos(\frac{2\pi}{3}) & -\sin(\frac{2\pi}{3}) \\ \sin(\frac{2\pi}{3}) & \cos(\frac{2\pi}{3}) \end{bmatrix} \begin{bmatrix} -\frac{1}{2}x + \frac{2\sqrt{3}}{4}y \\ -\frac{2\sqrt{3}}{4}x - \frac{1}{2}y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Finally, we have the original vector, so we see that a rotation about $\frac{2\pi}{3}$ corresponds to a rotation of order 3. If we apply this method for $q = 4$ and $q = 6$, we can see that rotation about $\frac{\pi}{2}$ and $\frac{\pi}{3}$ correspond to rotations of order 4 and 6, respectively.

Example 2: $q = 5$

For this example, we will go about things a little differently. Take the unit circle and divide it into 5 sections with a vector, v_i of unit length every $\frac{2\pi}{5}$ radians, with one vector pointing along the y -axis. Define $v_1 = v_2 = v_3 = v_4 = v_5$ to be the vectors of shortest length in the lattice. Since the set of lattice vectors is closed, the sum of any two lattice vectors must also be a lattice vector. We choose to add the vectors that are rotated $\frac{2\pi}{5}$ radians to the right

and left of $\frac{\pi}{2}$ to illustrate our point: $v_1 = \begin{bmatrix} \sqrt{\frac{5}{8}} + \frac{\sqrt{5}}{8} \\ \frac{1}{4}(\sqrt{5} - 1) \end{bmatrix}$ and $v_2 = \begin{bmatrix} -\sqrt{\frac{5}{8}} + \frac{\sqrt{5}}{8} \\ \frac{1}{4}(\sqrt{5} - 1) \end{bmatrix}$. It is easy

to see that when we add these two vectors together, we get $\begin{bmatrix} 0 \\ \frac{1}{2}(\sqrt{5} - 1) \end{bmatrix}$, which should be contained in J . However, $\frac{1}{2}(\sqrt{5} - 1) < 1$, a contradiction. We can extend this reasoning to show why a rotation in a wallpaper group cannot be of order 5.

Proof. Coming soon.

Islamic Art and the Seventeen Types of Symmetries

Again, there are a total of seventeen different wallpaper patterns, but we will only be focusing on the five that appear most frequently in Islamic art. The symbolic names for the groups each have a specific meaning. The first symbol describes the shape of the lattice structure and is always either a p (most cases) or a $c - p$ for “primitive” cell (a cell that contains the smallest generator of a pattern), c for “centered rectangular” cell, which is a rectangle repeat pattern imposed on a rhombus net for the sake of consistency. The number that follows indicates the highest order of rotational symmetry in the pattern and can be either 1, 2, 3, 4, or 6, as confirmed by the crystallographic restriction. The third and fourth symbols can either be m for “mirror reflection” or g for “glide reflection,” which we will not cover here.

Recalling the notation used above, we refer to G as the wallpaper group with translation subgroup H , point group J , and lattice L . \mathbf{a} and \mathbf{b} are vectors that span L , with \mathbf{b} being the vector of shortest length. A_θ rotates a vector around the origin by θ , while B_γ reflects a vector across a line through the origin with an angle of $\frac{\gamma}{2}$ to the x -axis. We now look at the five symmetry types present in Islamic art based on lattice shape (see figure 5 for a visual of what the patterns may look like).

Rectangular

A lattice is rectangular if $\|\mathbf{a}\| < \|\mathbf{b}\| < \|\mathbf{a} - \mathbf{b}\| = \|\mathbf{a} + \mathbf{b}\|$. There are four orthogonal transformations that preserve L : the identity, the rotation by π , reflection across the x -axis, and reflection across the y -axis, meaning that the point group will be a subgroup of $\{I, A_\pi = -I, B_0, B_\pi\}$.

pmm

The group p2mm is categorized by a rectangular lattice structure, 2-fold rotational symmetry, and two mirror lines. It can be shortened to just pmm because the presence of the two m's can adequately express the 2 without confusion. Since pmm contains both a vertical and horizontal reflection, the point group is the same as the general one for wallpaper groups with a rectangular lattice: $\{I, -I, B_0, B_\pi\}$.

Centered Rectangular

A lattice is rectangular if $\|\mathbf{a}\| < \|\mathbf{b}\| = \|\mathbf{a} - \mathbf{b}\| < \|\mathbf{a} + \mathbf{b}\|$. It is called *centered* rectangular because it is viewed as a rectangular lattice superimposed and centered over a rhombus lattice. The orthogonal transformations that preserve a centered rectangular lattice are the same ones that preserve a rectangular lattice.

cmm

The symmetry group c2mm has a centered rectangular lattice structure, 2-fold rotational symmetry, and two mirror lines. Similar to pmm, it can be shortened to cmm because the presence of the two mirror lines imply 2-fold rotational symmetry. In fact, cmm and pmm have the same the same point group as well. The difference between the wallpaper groups comes from slight differences in their lattice structure. Due to its rhombus-shaped background, the centered rectangular lattice has two rotational points and points of intersecting mirror lines. Thus cmm has two points of symmetry, while pmm has only one.

Square

A lattice is square if $\|\mathbf{a}\| = \|\mathbf{b}\| < \|\mathbf{a} - \mathbf{b}\| = \|\mathbf{a} + \mathbf{b}\|$. The points group that preserves L is the dihedral group of order 8, generated by $A_{\frac{\pi}{2}}$ and B_0 .

p4m

P4mm refers to the symmetry group of a pattern with a square lattice structure, 4-fold rotational symmetry, and mirror lines in two directions. As before, p4mm can be shortened to p4m. P4m is generated by $A_{\frac{\pi}{2}}$ and B_0 .

Hexagonal

A lattice is hexagonal if $\|\mathbf{a}\| = \|\mathbf{b}\| = \|\mathbf{a} - \mathbf{b}\| < \|\mathbf{a} + \mathbf{b}\|$. The point group is a subgroup of the dihedral group of order 12, generated by $A_{\frac{\pi}{3}}$ and B_0 .

p6m

P6mm describes the symmetry group of a pattern with a hexagonal lattice structure, 6-fold rotational symmetry, and mirror lines in two directions. This can be shortened to p6m because the 6-fold rotational symmetry and presence of one mirror line are enough to

categorize the symmetry group without causing confusion. It is generated by $A_{\frac{\pi}{3}}$ and B_0 , including both rotations and reflections.

p6

Lastly, patterns of type p6 have a hexagonal lattice structure and 6-fold rotational symmetry, but no reflectional symmetries. It is generated by solely $A_{\frac{\pi}{3}}$.

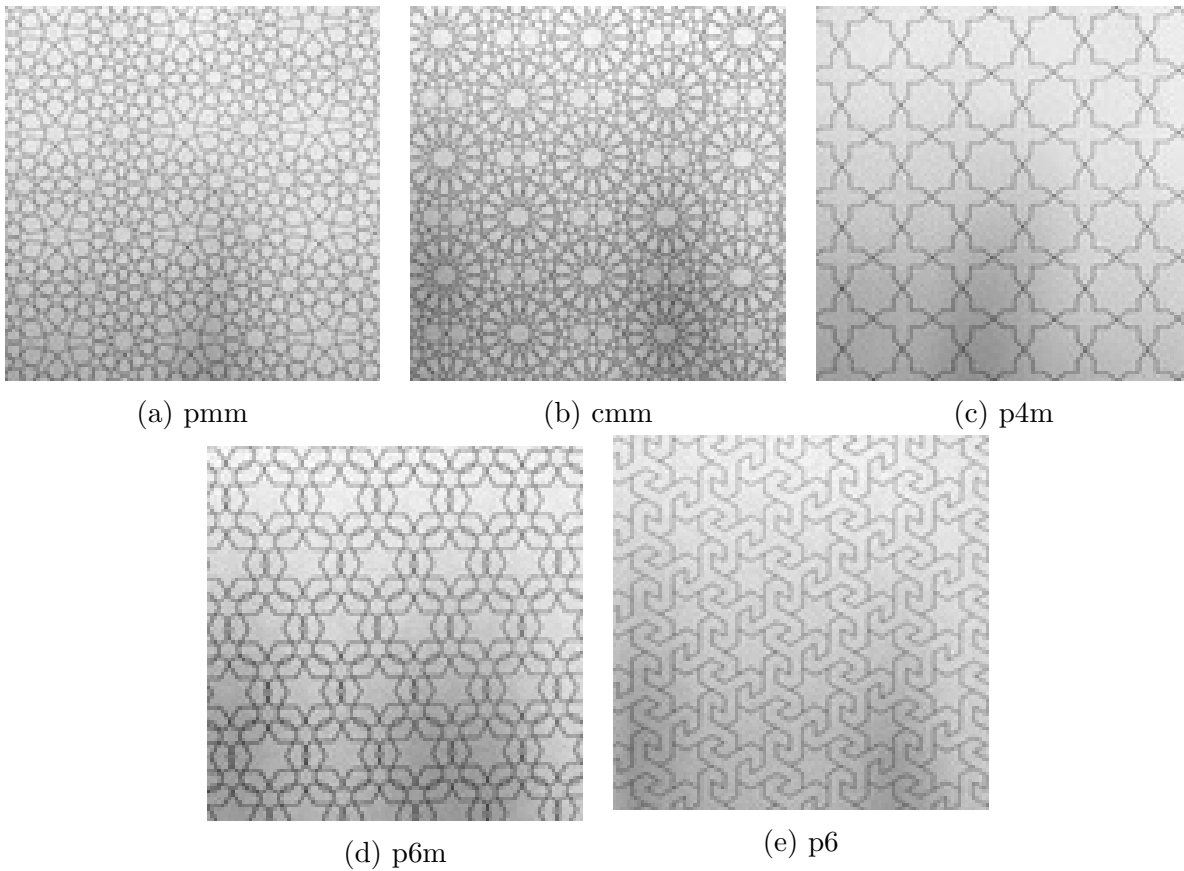


Figure 5: An example of each of the chosen wallpaper patterns

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