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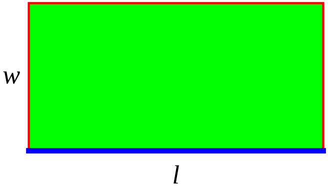

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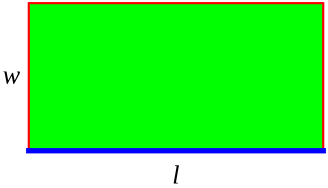

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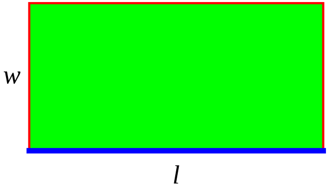

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Constraint: Need $\ell w = 1200$.

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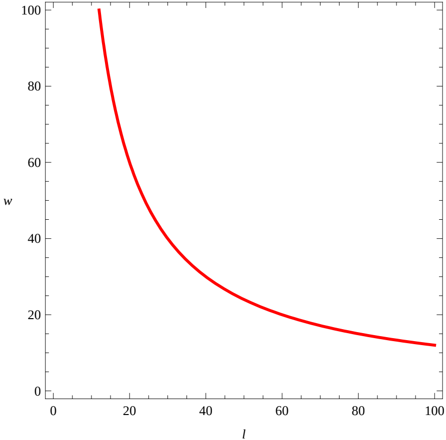
So build fence with expensive edge of length 30 meters and other dimension of 40 meters.

Idea for Method 2



Constraint curve $A = \ell w = 1200$

Level curves for objective $C = 80\ell + 60w$

Gradient vectors for constraint $A = \ell w$

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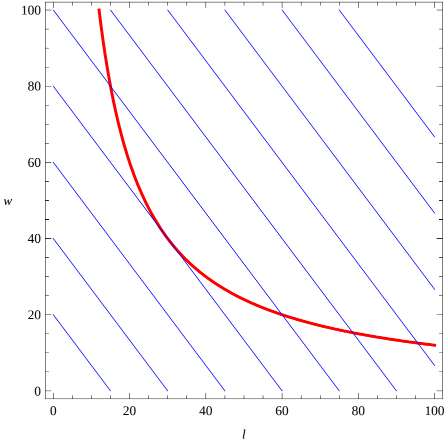

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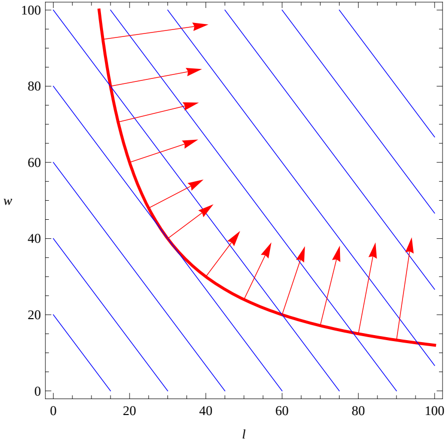

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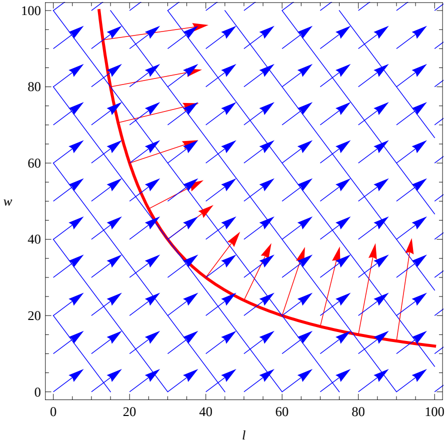

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$\vec{\nabla}(C + \lambda A) = \vec{0}$ for some constant λ