Integrating a vector field over a curve

Definition

We are given a vector field $\vec{F}$ and an oriented curve C in the domain of $\vec{F}$ as shown in the figure on the left below. The general idea of integrating the vector field $\vec{F}$ along the curve C is

add up over the curve infinitesimal contributions each having the form
(component of $\vec{F}$ tangent to C) times (length of piece of C).

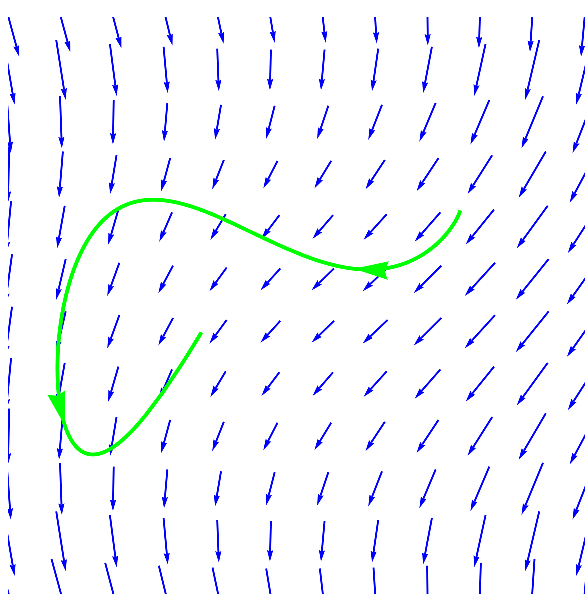
As usual, we will represent an infinitesimal displacement along the curve by $d\vec{r}$. A sample of these is shown as red vectors in the figure on the right below. The same figure also shows a sample vector field outputs along the curve (as blue vectors). For each point on the curve, there is an angle θ between the vector field output $\vec{F}$ and the infinitesimal displacement $d\vec{r}$ as shown in the below. In geometric terms, the dot product $\vec{F} \cdot d\vec{r}$ can be thought of as

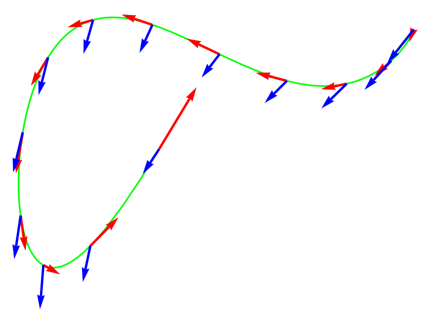
$$\vec{F} \cdot d\vec{r} = \|\vec{F}\| \|d\vec{r}\| \cos\theta = \|\vec{F}\| ds \ \cos\theta = \left(\|\vec{F}\| \cos\theta\right) ds.$$

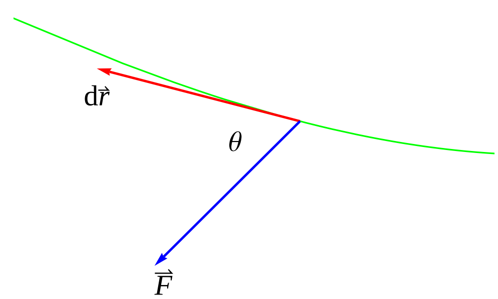
Note that $\|\vec{F}\| \cos\theta$ is the component of $\vec{F}$ tangent to the curve and ds is the length of an infinitesimal piece of the curve. So, we can think of the integration as

add up over the curve infinitesimal contributions each having the form $\vec{F} \cdot d\vec{r}$.

We will denote this type of integral as $\displaystyle\int_C \vec{F} \cdot d\vec{r}$.







Names

An integral of this type is commonly called a *line integral for a vector field*. This name is a bit misleading since the curve C need not be a line. Other names in use include *curve integral*, *work integral*, and *flow integral*. Also note that we need to distinguish between

$$\int\limits_C f\,ds \qquad \text{and} \qquad \int\limits_C \vec{F}\cdot d\vec{r}.$$

The first of these is a line integral for a *scalar field* f while the second is a line integral for a *vector field* $\vec{F}$.

Notation

The text often uses an alternate notation for the line integral. Here's the connection: We can denote the magnitude of the (infinitesimal) displacement vector $d\vec{r}$ by $ds = \|d\vec{r}\|$. The displacement vector $d\vec{r}$ is tangent to the curve (at a particular point), so we can denote the direction by $\vec{T}$. That is, $\vec{T}$ is a unit vector tangent to the curve (at a point). In terms of magnitude and direction, we can write $d\vec{r} = \vec{T}\,ds$. With this, we can denote a line integral by

$$\int\limits_C \vec{F}\cdot d\vec{r} \qquad \text{or} \qquad \int\limits_C \vec{F}\cdot\vec{T}\,ds.$$

I generally use the first notation while the text favors the second.

There is another very common notation in use. Here's the connection: Write the vector field $\vec{F}$ in terms of components as $\vec{F} = M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}$ and write the vector $d\vec{r}$ in terms of components as $d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$. We think of dx as a small displacement parallel to the x-axis, dy as a small displacement parallel to the y-axis, and dz as a small displacement parallel to the z-axis. With these component expressions, we can write out the dot product as

$$\vec{F}\cdot d\vec{r} = \left(M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}\right)\cdot\left(dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}\right) = M\,dx + N\,dy + P\,dz.$$

Using this, the notation for line integral can be written

$$\int\limits_C \vec{F}\cdot d\vec{r} = \int\limits_C M\,dx + N\,dy + P\,dz.$$

Some texts favors the expression on the right side and I generally use the expression on the left side.

Some of the problems are given using the notation on right side. For example, Problem 27 of Section 16.2 gives the line integral

$$\int_C (x-y)\,dx + (y-z)\,dy + z\,dz.$$

From this, you can read off that the vector field is $\vec{F} = (x-y)\,\hat{\imath} + (y-z)\,\hat{\jmath} + z\hat{k}$.

Computing line integrals

In computing line integrals, the general plan is to express everything in terms of a single variable. This is a reasonable thing to do because a curve is a one-dimensional object. The essential things are to determine the form of $d\vec{r}$ for the curve C and the outputs $\vec{F}$ along the curve C, all in terms of one variable. The displacement $d\vec{r}$ is defined to have components

$$d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath}$$

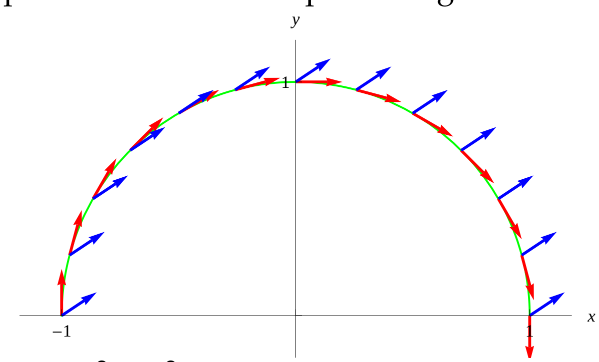
How to proceed depends on how we describe the curve. In general, we have two choices: a relation between the coordinates or a parametric description. The two solutions to the following example show how to work with each of these.

Example

Compute the line integral of $\vec{F}(x, y) = 3\,\hat{\imath} + 2\,\hat{\jmath}$ *for the curve C that is the upper half of the circle of radius* 1 *traversed from left to right.*

Solution 1:

We start by making a plot showing the curve, a sample of infinitesimal displacements, and a sample of vector field outputs along the curve as shown below.



The equation of the circle is $x^2 + y^2 = 1$. From this, we compute

$$2x\,dx + 2y\,dy = 0.$$

Solving for dy and substituting from $x^2 + y^2 = 1$ gives

$$dy = -\frac{x}{y}\,dx = -\frac{x}{\sqrt{1-x^2}}\,dx.$$

This is the relation between dx and dy for a displacement $d\vec{r}$ along the circle. Substituting this gives

$$d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath} = dx\,\hat{\imath} - \frac{x}{\sqrt{1-x^2}}\,dx\,\hat{\jmath} = \left(\hat{\imath} - \frac{x}{\sqrt{1-x^2}}\,\hat{\jmath}\right) dx$$

(Note that this expression is undefined for $x = \pm 1$. This will not affect the value we compute for the integral because this is a finite set of points and not an interval of values. The method below avoids this issue.)

The vector field here is constant so all outputs along the curve C are $\vec{F} = 3\hat{\imath} + 2\hat{\jmath}$. We thus have

$$\vec{F} \cdot d\vec{r} = \left(3\,\hat{\imath} + 2\,\hat{\jmath}\right) \cdot \left(\hat{\imath} - \frac{x}{\sqrt{1-x^2}}\,\hat{\jmath}\right) dx = \left(3 - \frac{2x}{\sqrt{1-x^2}}\right) dx$$

This is the integrand. For the curve C, the variable x ranges from -1 to 1, so we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_{-1}^{1} \left(3 - \frac{2x}{\sqrt{1-x^2}}\right) dx = \text{ some work to be done here } = 6.$$

Solution 2:

We use polar coordinates to write

$$x = \cos\theta \qquad \text{and} \qquad y = \sin\theta \qquad \text{for } \theta \text{ from } \pi \text{ to } 0.$$

Note that having θ range from π to 0 that this traces out the curve C in the correct direction (from left to right). From these, we compute

$$dx = -\sin\theta\, d\theta \qquad \text{and} \qquad dy = \cos\theta\, d\theta.$$

Substituting into $d\vec{r}$ gives

$$d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath} = -\sin\theta\, d\theta\,\hat{\imath} + \cos\theta\, d\theta\hat{\jmath} = \left(-\sin\theta\,\hat{\imath} + \cos\theta\hat{\jmath}\right) d\theta.$$

The vector field here is constant so all outputs along the curve C are $\vec{F} = 3\,\hat{\imath} + 2\,\hat{\jmath}$. We thus have

$$\vec{F} \cdot d\vec{r} = \left(3\,\hat{\imath} + 2\,\hat{\jmath}\right) \cdot \left(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}\right) d\theta = \left(-3\sin\theta + 2\cos\theta\right) d\theta.$$

This is the integrand. For the curve C, the variable θ ranges from π to 0, so we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_{\pi}^{0} \left(-3\sin\theta + 2\cos\theta\right) dt = \text{some work to be done here } = 6.$$

More on notation

For many problems in the text, the curve C is given as a function

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} \qquad \text{for a curve in the plane}$$

or

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k} \qquad \text{for a curve in space.}$$

For example, Problem 9 involves a curve C_2 given by

$$\vec{r}(t) = t\,\hat{\imath} + t^2\,\hat{\jmath} + t^4\,\hat{k} \qquad \text{for } 0 \le t \le 1$$

We can read off from this that x, y, and z for points on the curve are given by

$$x = t, \qquad y = t^2, \qquad \text{and} \qquad z = t^4 \qquad \text{for } 0 \le t \le 1$$

To get $d\vec{r}$ in terms of the variable t, we compute

$$dx = dt, \qquad dy = 2t\,dt, \qquad \text{and} \qquad dz = 4t^3\,dt$$

and then substitute to get

$$d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k} = dt\,\hat{\imath} + 2t\,dt\,\hat{\jmath} + 4t^3\,dt\,\hat{k} = \left(\hat{\imath} + 2t\ \hat{\jmath} + 4t^3\ \hat{k}\right)dt$$

Note that we could also get this by directly "d-ing" both sides of

$$\vec{r}(t) = t\,\hat{\imath} + t^2\,\hat{\jmath} + t^4\,\hat{k}.$$

Problems: Integrating a vector field over a curve

For each of the following,

- Sketch the given vector field $\vec{F}$ and the given curve C.
- Use your sketch to determine or estimate the sign of $\displaystyle\int_C \vec{F} \cdot d\vec{r}$.
- Compute the value of $\displaystyle\int_C \vec{F} \cdot d\vec{r}$.

1. $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath}$
   C is the semicircle of radius 1 from $(-1, 0)$ to $(1, 0)$ with $y \ge 0$

   Answer: 0

2. $\vec{F} = y\,\hat{\imath} - x\,\hat{\jmath}$
   C is the semicircle of radius 1 from $(-1, 0)$ to $(1, 0)$ with $y \ge 0$

   Answer: π

3. $\vec{F} = y\,\hat{\imath} + x\,\hat{\jmath}$
   C is given by $x = t^3$ and $y = t^2$ for $t = 0$ to $t = 2$

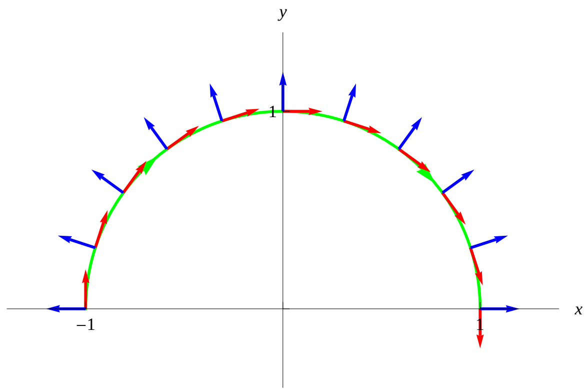
   Answer: 32

4. $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$
   C is the helix with constant pitch wrapping 5 times around a (right circular) cylinder of radius 2 and height 20

   Answer: 200

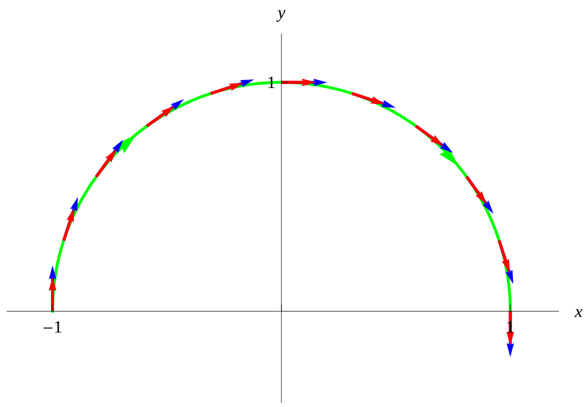
Plots and a solution for
Problems: Integrating a vector field over a curve

1. $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath}$
   C is the semicircle of radius 1 from $(-1, 0)$ to $(1, 0)$ with $y \geq 0$



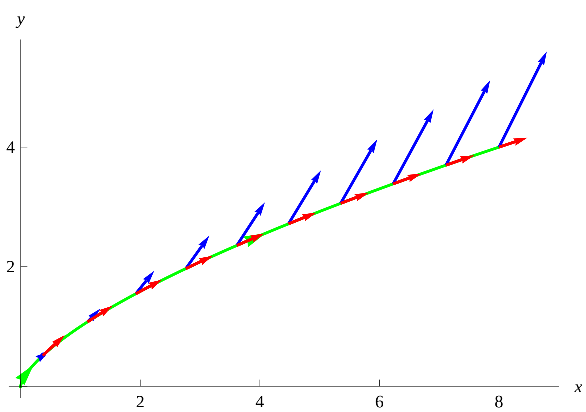
Answer: 0

2. $\vec{F} = y\,\hat{\imath} - x\,\hat{\jmath}$
   C is the semicircle of radius 1 from $(-1, 0)$ to $(1, 0)$ with $y \geq 0$



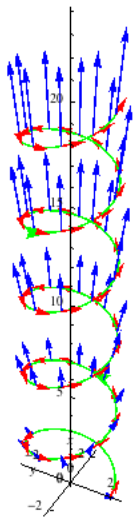
Answer: π

3. $\vec{F} = y\,\hat{\imath} + x\,\hat{\jmath}$
   C is given by $x = t^3$ and $y = t^2$ for $t = 0$ to $t = 2$



Answer: 32

4. $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$
   C is the helix with constant pitch wrapping 5 times around a (right circular) cylinder of radius 2 and height 20



Solution:

We can describe the helix using cylindrical coordinates with $r = 2$ to get

$$x = 2\cos\theta \qquad y = 2\sin\theta \qquad z = \frac{20}{10\pi}\theta = \frac{2}{\pi}\theta$$

for $0 \leq \theta \leq 2\pi$. Note that the helix having constant pitch means that z is proportional to *theta*; the proportionality constant is determined by the requirement that helix goes up 20 units in 5 wraps. From these, we compute

$$dx = -2\sin\theta\,d\theta \qquad dy = 2\cos\theta\,d\theta \qquad dz = \frac{2}{\pi}\,d\theta$$

to get

$$d\vec{r} = \left(-2\sin\theta\,\hat{\imath} + 2\cos\theta\,\hat{\jmath} + \frac{2}{\pi}\,\hat{k}\right)d\theta.$$

Along the curve, the vector field is

$$\vec{F} = 2\cos\theta\,\hat{\imath} + 2\sin\theta\,\hat{\jmath} + \frac{2}{\pi}\theta\,\hat{k}.$$

Dotting these together gives us

$$\vec{F}\cdot d\vec{r} = \left(-4\sin\theta\cos\theta + 4\cos\theta\sin\theta + \frac{4}{\pi^2}\theta\right)d\theta = \frac{4}{\pi^2}\theta\,d\theta.$$

Putting together the details, we get

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{10\pi} \frac{4}{\pi^2}\theta\,d\theta = \frac{4}{\pi^2}\frac{\theta^2}{2}\bigg|_0^{10\pi} = 200.$$