Integration over a surface

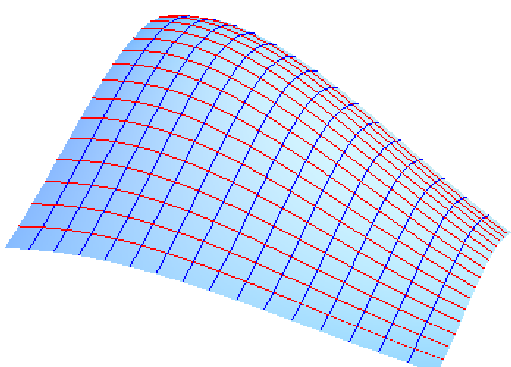
Given a surface S in space, we can (conceptually) break it into small pieces each of which has area dA. In some cases, we will add up these small contributions to get the total area of the curve. We will represent this as

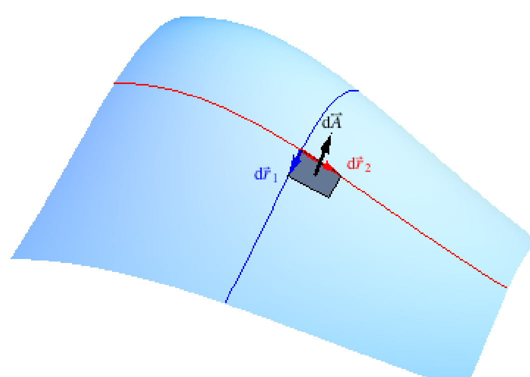
$$A = \iint\limits_S dA.$$

In other cases, we will have an area density σ defined at each point on the surface and we will add up small contributions of the form $\sigma\, dA$ to get a total (of some quantity such as charge or mass). We will represent this as

$$\text{Total} = \iint\limits_S \sigma\, dA.$$

To evaluate a given surface integral, we will generally build an iterated integral in two variables. We can think of each of these variables as defining a family of curves that fills the surface so that, at each point on the surface, a curve from one family intersects a curve from the other family non-tangentially. An example is shown in the figure below on the left with one family of curves in blue and the other in red. Focus on a generic point on the surface. At this point, we can consider the infinitesimal displacement $d\vec{r}_1$ along one curve and the infinitesimal displacement $d\vec{r}_2$ along the other curve as shown in the figure on the right below. These two infinitesimal displacements form a parallelogram that has infinitesimal area vector $d\vec{A} = d\vec{r}_1 \times d\vec{r}_2$. The magnitude of $d\vec{A}$ is our area element. That is, $dA = \|d\vec{A}\|$.





As with curve integrals, how we compute $d\vec{r}_1$ and $d\vec{r}_2$, (in order to compute $d\vec{A} = d\vec{r}_1 \times d\vec{r}_2$, and $dA = \|d\vec{A}\|$) depends on how we choose to describe the surface. Various approaches are best illustrated by examples. In the first two examples, we will take different approaches to the same problem.

Example 1

Compute the area of the section of the paraboloid $z = x^2 + y^2$ for $0 \le z \le 4$.

The area is given by

$$A = \iint\limits_{\text{paraboloid}} dA.$$

We will evaluate this integral by constructing an iterated integral in two variables and then evaluating that iterated integral. In this example, we make direct use of the relation $z = x^2 + y^2$ among the cartesian coordinates x, y, and z.

One family of curves on the surface is given by taking y to be constant. These curves are parabolas in planes parallel to the xz-plane shown as the blue curves in the figure below. From $z = x^2 + y^2$ where y is a constant, we compute

$$dz = 2x\,dx \qquad \text{and} \qquad dy = 0.$$

Substituting into $d\vec{r}_1 = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$, we get

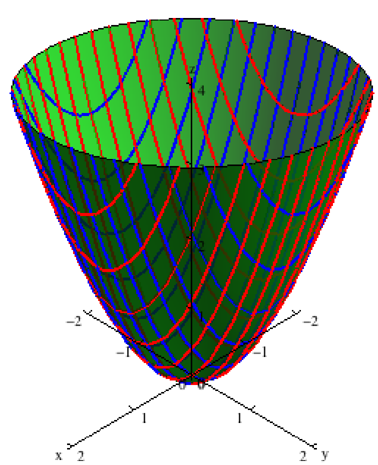
$$d\vec{r}_1 = dx\,\hat{\imath} + 0\,\hat{\jmath} + 2x\,dx\,\hat{k} = (\hat{\imath} + 2x\,\hat{k})\,dx$$

A second family of curves on the surface is given by taking x to be constant. These curves are parabolas in planes parallel to the yz-plane shown as the red curves in the figure below. From $z = x^2 + y^2$ where x is a constant, we compute

$$dz = 2y\,dy \qquad \text{and} \qquad dx = 0.$$

Substituting into $d\vec{r}_2 = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$, we get

$$d\vec{r}_2 = 0\,\hat{\imath} + dy\,\hat{\jmath} + 2y\,dy\,\hat{k} = (\hat{\jmath} + 2y\,\hat{k})\,dy$$



We now compute $d\vec{A}$ as a cross product of $d\vec{r}_1$ and $d\vec{r}_2$:

$$\begin{aligned} d\vec{A} &= d\vec{r}_1 \times d\vec{r}_2 \\ &= (\hat{\imath} + 2x\,\hat{k})\,dx \;\times\; (\hat{\jmath} + 2y\,\hat{k})\,dy \\ &= (\hat{\imath} + 2x\,\hat{k}) \times (\hat{\jmath} + 2y\,\hat{k})\,dxdy \\ &= (\hat{\imath} \times \hat{\jmath}) + 2y\,(\hat{\imath} \times \hat{k}) + 2x\,(\hat{k} \times \hat{\jmath}) + 4xy\,(\hat{k} \times \hat{k}) \\ &= \hat{k} + 2y(-\hat{\jmath}) + 2x(-\hat{\imath}) + 4xy(\hat{0}) \\ &= -2x\,\hat{\imath} - 2y\,\hat{\jmath} + \hat{k}. \end{aligned}$$

We next compute the magnitude to get

$$dA = \|d\vec{A}\| = \sqrt{4x^2 + 4y^2 + 1}\,|dx||dy|.$$

To put together an iterated integral, we need to determine bounds on the variables x and y. In this case, the paraboloid projects onto the disk of radius 2 in the xy-plane so we can use the bounds

$$-2 \le x \le 2 \qquad \text{and} \qquad -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}.$$

With these and our expression for dA, we have that the area is given by

$$A = \iint\limits_{\text{paraboloid}} dA = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{4x^2 + 4y^2 + 1}\,dxdy.$$

Perhaps the easiest way to evaluate this integral is to convert to polar coordinates which leads to

$$\begin{aligned} A = \iint\limits_{\text{paraboloid}} dA &= \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1}\,r\,drd\theta \\ &= \int_0^{2\pi} d\theta \int_0^2 \sqrt{4r^2 + 1}\,r\,dr \\ &= (2\pi) \left[\frac{1}{8} \frac{2}{3} (4r^2 + 1)^{3/2} \right|_0^2 \qquad \text{using a substitution } u = 4r^2 + 1 \\ &= \frac{(17^{3/2} - 1)\pi}{6} \approx 11.52\pi. \end{aligned}$$

To check if this is reasonable, we can compare to the surface area of a cylinder of radius 2 and height 4 that includes the lateral side and a bottom but not a top. (You should visualize or draw a picture to see the relationship between the paraboloid and this cylinder.) This has an area of $2\pi(2)(4) + \pi(2)^2 = 12\pi$. The smaller value for the area of the paraboloid is consistent with this.

In our next approach to this same problem, we will use cylindrical coordinates so we will get to the polar coordinate integral in a natural way.

Example 2

Compute the area of the section of the paraboloid $z = x^2 + y^2$ *for* $0 \le z \le 4$.

In cylindrical coordinates, the equation of the paraboloid is $z = r^2$. With this, we can express x, y, and z in terms of the two variables r and θ as

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad \text{and} \qquad z = r^2.$$

We can get two families of curves by taking θ constant (blues curves in the plot below) and by taking r constant (red curves in the plot below). For the first family of curves with θ constant, we have

$$dx = \cos\theta\, dr, \qquad dy = \sin\theta\, dr, \qquad \text{and} \qquad dz = 2r\, dr$$

so

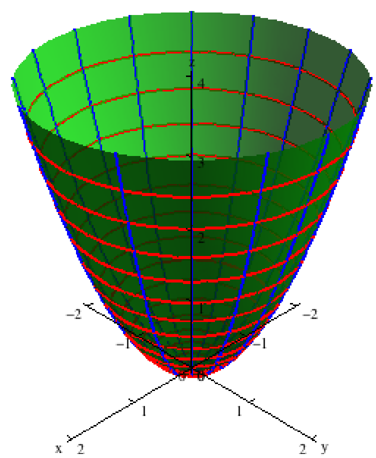
$$d\vec{r}_1 = \cos\theta\, dr\, \hat{\imath} + \sin\theta\, dr\, \hat{\jmath} + 2r\, dr\, \hat{k} = (\cos\theta\, \hat{\imath} + \sin\theta\, \hat{\jmath} + 2r\, \hat{k})\, dr.$$

For the second family of curves with r constant, we have

$$dx = -r\sin\theta\, d\theta, \qquad dy = r\cos\theta\, d\theta, \qquad \text{and} \qquad dz = 0$$

so

$$d\vec{r}_2 = -r\sin\theta\, d\theta\, \hat{\imath} + r\cos\theta\, d\theta\, \hat{\jmath} + 0\, \hat{k} = (-r\sin\theta\, \hat{\imath} + r\cos\theta\, \hat{\jmath})\, d\theta.$$



We now compute $d\vec{A}$ as a cross product of $d\vec{r}_1$ and $d\vec{r}_2$:

$$\begin{aligned}
d\vec{A} &= d\vec{r}_1 \times d\vec{r}_2 \\
&= (\cos\theta\, \hat{\imath} + \sin\theta\, \hat{\jmath} + 2r\, \hat{k})\, dr \;\times\; (-r\sin\theta\, \hat{\imath} + r\cos\theta\, \hat{\jmath})\, d\theta \\
&= r\cos^2\theta\, (\hat{\imath} \times \hat{\jmath}) - r\sin^2\theta\, (\hat{\jmath} \times \hat{\imath}) - 2r^2\sin\theta\, (\hat{k} \times \hat{\imath}) + 2r^2\cos\theta\, (\hat{k} \times \hat{\jmath}) \\
&= r\cos^2\theta\, (\hat{k}) - r\sin^2\theta\, (-\hat{k}) - 2r^2\sin\theta\, (\hat{\jmath}) + 2r^2\cos\theta\, (-\hat{\imath}) \\
&= -2r^2\cos\theta\, \hat{\imath} - 2r^2\sin\theta\, \hat{\jmath} + r(\cos^2\theta + \sin^2\theta)\, \hat{k} \\
&= -2r^2\cos\theta\, \hat{\imath} - 2r^2\sin\theta\, \hat{\jmath} + r\, \hat{k}.
\end{aligned}$$

We next compute the magnitude to get

$$dA = \|d\vec{A}\| = \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2}\,|dr||d\theta| = r\sqrt{4r^2+1}\,|dr||d\theta|.$$

To put together an iterated integral, we need to determine bounds on the variables r and θ. In this case, the paraboloid projects onto the disk of radius 2 in the $r\theta$-plane so we can use the bounds

$$0 \le \theta \le 2\pi \qquad \text{and} \qquad 0 \le r \le 2.$$

With these and our expression for dA, we have that the area is given by

$$A = \iint_{\text{paraboloid}} dA = \int_0^{2\pi}\int_0^2 r\sqrt{4r^2+1}\,dr d\theta.$$

Note that this is precisely the same integral we evaluated in Example 1 so we can follow the same steps to get

$$A = \frac{(17^{3/2}-1)\pi}{6} \approx 11.52\pi.$$

We have already checked that this result is reasonable by comparing with an easy-to-compute area.

We now look at an example of computing a total from a density.

Example 3

Charge is distributed on a hemisphere of radius R with area charge density proportional to the angle from the "north pole" of the hemisphere. Compute the total charge Q in terms of R and the maximum density σ_0.

We will compute the total charge as

$$Q = \iint_{\text{hemisphere}} \sigma\, dA$$

where σ is the area charge density.

We can use spherical coordinates with $\rho = R$ to write

$$x = R\sin\phi\cos\theta, \qquad y = R\sin\phi\sin\theta, \qquad \text{and} \qquad z = R\cos\phi$$

for $0 \le \phi \le \pi/2$ and $0 \le \theta \le 2\pi$. We can get two families of curves by taking θ constant (latitude circles on the sphere) and by taking ϕ constant (longitude semicircles on the sphere). For the first family of curves with θ constant, we have

$$dx = R\cos\phi\cos\theta\,d\phi, \qquad dy = R\cos\phi\sin\theta\,d\phi, \qquad \text{and} \qquad dz = -R\sin\phi\,d\phi$$

so

$$d\vec{r}_1 = R(\cos\phi\cos\theta\,\hat{\imath} + \cos\phi\sin\theta\,\hat{\jmath} - \sin\phi\,\hat{k})\,d\phi.$$

For the second family of curves with ϕ constant, we have

$$dx = -R\sin\phi\sin\theta\, d\theta, \qquad dy = R\sin\phi\cos\theta\, d\theta, \qquad \text{and} \qquad dz = 0$$

so

$$d\vec{r}_2 = R(\sin\phi\sin\theta\,\hat{\imath} + \sin\phi\cos\theta\,\hat{\jmath})\, d\theta.$$

Computing $d\vec{A} = d\vec{r}_1 \times d\vec{r}_2$ and then computing the magnitude, we get

$$dA = R^2 \sin\phi\, d\phi d\theta.$$

(If you have not already done so in a previous example or problem, you should fill in the details of these calculations. As an alternative, we could make a geometric argument to get this expression for dA. We have previously done this type of argument to get the volume element expression $dV = \rho^2 \sin\phi\, d\rho d\phi d\theta$.)

The charge density σ is proportional to ϕ so we have $\sigma = k\phi$ for some proportionality constant k. The charge density will have its maximum value σ_0 for $\phi = \pi/2$ so

$$\sigma_0 = k\frac{\pi}{2} \qquad \text{which implies} \qquad k = \frac{2\sigma_0}{\pi}.$$

Thus, the area density σ is related to ϕ by

$$\sigma = \frac{2\sigma_0}{\pi}\phi.$$

So, we can express the total charge as

$$Q = \iint\limits_{\text{hemisphere}} \sigma\, dA = \int_0^{2\pi}\int_0^{\pi/2} \frac{2\sigma_0}{\pi}\phi\, R^2 \sin\phi\, d\phi d\theta.$$

The iterated integral is easy to evaluate by factoring and using the Fundamental Theorem of Calculus, giving us

$$Q = \frac{2\sigma_0}{\pi}R^2 \int_0^{2\pi} d\theta \int_0^{\pi/2} \phi\sin\phi\, d\phi = \frac{2\sigma_0}{\pi}R^2(2\pi)\left[\sin\phi - \phi\cos\phi\right|_0^{\pi/2} = \frac{2\sigma_0}{\pi}R^2(2\pi)(1)$$
$$= 4R^2\sigma_0.$$

Note that our result $Q = 4R^2\sigma_0$ has the correct units. We can also check that it is reasonable by comparing to some easy-to-compute quantity. Specifically, for a hemisphere with a uniform charge density of σ_0 at each point, the total charge is $2\pi R^2\sigma_0$. Our result of $4R^2\sigma_0$ is less than this which is consistent with having charge density less than σ_0 at points other than the equator of the hemisphere.

Problems: Integration over a surface

1. Compute the surface area of a sphere of radius R

   Answer: $A = 4\pi R^2$

2. Compute the surface area of the lateral side of a right circular cone of height H and radius R.

   Answer: $A = \pi R\sqrt{R^2 + H^2}$

3. A *torus* is the doughnut-shaped surface formed by bending and gluing a right circular cylinder section so that the central axis forms a circle. (You can also think about a torus as generated by revolving a circle around a fixed axis that does not intersect the circle.) Let R be the radius of the cylinder and $2\pi B$ be the height of the cylinder. (Equivalently, R is the radius of the circle and B is the distance from the circle center to the rotation axis.) The cartesian coordinates of points on a torus can be described by the equations

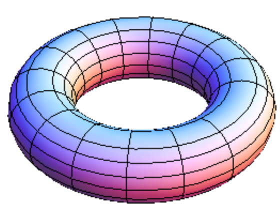
   $$x = (B + R\sin\phi)\cos\theta, \qquad y = (B + R\sin\phi)\sin\theta, \qquad \text{and} \qquad z = R\cos\phi.$$

   for $0 \le \phi \le 2\pi$ and $0 \le \theta \le 2\pi$. Note that we need $B > R$ to have a true torus.

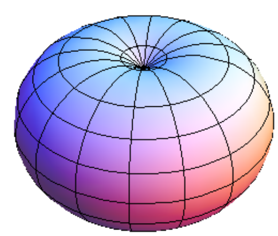
   Compute the surface area of a torus with dimensions B and R with $B > R$.

   Answer: $A = 4\pi^2 RB$

   Optional challenge problem: Analyze the case $B < R$.



$B = 3$ and $R = 1$



$B = 2$ and $R = 3$

4. Charge is distributed on a hemisphere of radius R so that the area charge density is proportional to the distance from the equatorial plane. Compute the total charge Q in terms of R and the maximum density σ_0.

5. Charge is distributed on a torus generated by rotating a circle of radius R around a circle of radius B (with $B > R$) so that the area charge density is proportional to the distance from the central axis. Compute the total charge Q in terms of R, B, and the maximum density σ_0.

   Answer: $Q = 2\pi^2 \dfrac{R(2B^2 + R^2)}{B + R}\sigma_0 = 4\pi^2 RB\dfrac{2B^2 + R^2}{2B^2 + 2R^2}\sigma_0$