

### More on equations of planes

So far, we have seen several forms for the equation of a plane:

$$\begin{array}{ll} Ax + By + Cz + D = 0 & \text{standard form} \\ z = m_x x + m_y y + b & \text{slopes-intercept form} \\ z - z_0 = m_x(x - x_0) + m_y(y - y_0) & \text{point-slopes form} \end{array}$$

Using vectors, we can add another form that is coordinate-free.

A plane can be specified by giving a vector  $\vec{n}$  perpendicular to the plane (called a *normal vector*) and a point  $P_0$  on the plane. We can develop a condition or test to determine whether or not a variable point  $P$  is on the plane by thinking geometrically and using the dot product. Here's the reasoning:

- $P$  is on the plane if and only if the vector  $\overrightarrow{P_0P}$  is parallel to the plane.
- The vector  $\overrightarrow{P_0P}$  is parallel to the plane if and only if  $\overrightarrow{P_0P}$  is perpendicular to the normal vector  $\vec{n}$ .
- The vectors  $\overrightarrow{P_0P}$  and  $\vec{n}$  are perpendicular if and only if their dot product is zero:

$$\vec{n} \cdot \overrightarrow{P_0P} = 0.$$

So, the condition  $\vec{n} \cdot \overrightarrow{P_0P} = 0$  is a new form for the equation of a plane. We'll refer to this as the *point-normal form*. We can see how the point-normal form relates to our familiar forms by introducing coordinates and components. Let  $P_0$  have coordinates  $(x_0, y_0, z_0)$ , the variable point  $P$  have coordinates  $(x, y, z)$ , and the normal vector  $\vec{n}$  have components  $\langle n_x, n_y, n_z \rangle$ . With these, the vector  $\overrightarrow{P_0P}$  has components  $\langle x - x_0, y - y_0, z - z_0 \rangle$ . So, the point-normal form can be written as

$$\begin{aligned} 0 &= \vec{n} \cdot \overrightarrow{P_0P} \\ &= \langle n_x, n_y, n_z \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle \\ &= n_x(x - x_0) + n_y(y - y_0) + n_z(z - z_0) \\ &= n_x x + n_y y + n_z z - (n_x x_0 + n_y y_0 + n_z z_0). \end{aligned}$$

The last expression is the same as  $Ax + By + Cz + D$  if we identify  $n_x$  as  $A$ ,  $n_y$  as  $B$ ,  $n_z$  as  $C$  and  $-(n_x x_0 + n_y y_0 + n_z z_0)$  as  $D$ . This is perhaps easier to see in an example.

#### Example

Find the standard form for the equation of the plane that contains the point  $(6, 5, 2)$  and has normal vector  $\langle 7, -3, 4 \rangle$ .

With  $(x, y, z)$  as the coordinates of a variable point, we can write

$$\begin{aligned} 0 &= \vec{n} \cdot \overrightarrow{P_0P} \\ &= \langle 7, -3, 4 \rangle \cdot \langle x - 6, y - 5, z - 2 \rangle \\ &= 7(x - 6) - 3(y - 5) + 4(z - 2) \\ &= 7x - 3y + 4z - 42 + 15 - 8 \\ &= 7x - 3y + 4z - 35. \end{aligned}$$

So the standard form of the equation for this plane is  $7x - 3y + 4z - 35 = 0$ .

## Exercises

1. Use the point-normal equation to determine which, if any, of the following points are on the plane that has normal vector  $2\hat{i} - \hat{j} + 6\hat{k}$  and contains the point  $(3, 4, 2)$ .

(a)  $(5, -4, 0)$

(b)  $(1, 6, 2)$

(c)  $(2, 8, 3)$

*Answer:*  $(5, -4, 0)$  and  $(2, 8, 3)$  are on the plane,  $(1, 6, 2)$  is not

2. Find the slopes-intercept form of the equation that contains the point  $(4, 2, -7)$  and has normal vector  $\vec{n} = 5\hat{i} - 3\hat{j} + 2\hat{k}$ .

*Answer:*  $z = -\frac{5}{2}x + \frac{3}{2}y$

3. Find the slopes-intercept form of the equation for the plane that contains the point  $(4, 2, -7)$  and has normal vector  $\vec{n} = \langle -6, 1, 5 \rangle$ .

*Answer:*  $z = \frac{6}{5}x - \frac{1}{5}y - \frac{57}{5}$

4. Find the standard form of the equation for the plane that contains the point  $(6, 3, 0)$  and is parallel to a second plane given by the equation  $5x + 2y - 9z = 14$ .

5. Find the standard form of the equation for the plane that contains the point  $(7, -2, 1)$  and is perpendicular to the vector from the origin to that same point.

*Answer:*  $7x - 2y + z - 54 = 0$