Problems on differentials

1. The volume V of a right circular cylinder is related to the radius r and height h of the cylinder by $V = \pi r^2 h$.
   (a) Find the linear relation among the differentials dV, dr, and dh.
   Answer: $dV = \pi r^2\, dh + 2\pi rh\, dr$
   (b) Use your result from (a) to deduce a relation among percent changes in V, r, and h.
   Answer: $\dfrac{dV}{V} = \dfrac{dh}{h} + 2\dfrac{dr}{r}$
   (c) If the height and radius of a cylinder are each increased by 1%, by what percent does the volume increase?
   Answer: 3%
   (d) If the height of a cylinder is increased by 1%, how must the radius be changed to keep volume constant?
   Answer: Decrease radius by 1/2%

2. Consider a consumer who can purchase different amounts of three commodities: apples, bananas, and cherries. Let a, b, and c be the amount purchased of each (measured in pounds). A simple model used by economists assigns a utility U (in units we'll call *utils*) to each bundle (a, b, c) the consumer can purchase according to the formula

$$U = k\, a^{1/2} b^{1/6} c^{1/3}$$

   where $k = 1$ util/lb (to keep units consistent).
   (a) Find the linear relation among differentials dU, da, db, and dc.
   (b) Use your result from (a) to deduce a relation among percent changes in U, a, b, and c.
   (c) For which of the commodities would 1% increase in amount purchased lead to the smallest change in utility? What is the percentage change in utililty corresponding to a 1% increase in the amount purchased of that commodity?

3. The volume V of a sphere is related to the radius r of the sphere by $V = \frac{4}{3}\pi r^3$.
   (a) Find the linear relation between the differentials dV and dr.
   (b) Suppose volume and radius are changing in time t. Use your result from (a) to get a relation between the rate of change in V with respect to t and the rate of change in r with respect to t.
   (c) Suppose air is being pumped into a balloon at the rate 0.2 cubic meters per second. How fast is the radius changing at the time when the radius is 1.5 meters?

There is more on the flip side.

4. Consider the relation $z = \cos(xy)$.

   (a) Find the linear relation among the differentials dx, dy, and dz.

   (b) Consider a level curve in the xy-plane for z constant so $dz = 0$. Use your relation from (a) to get a formula for the slope dy/dx of a level curve.

   (c) Use your result in (b) to compute the slope of the level curve that passes through the point $(x, y) = (5, 2)$.

5. Suppose x, y, and z are related by a function $z = f(x, y)$.

   (a) Find the linear relation among the differentials dx, dy, and dz. Note that this will involve partial derivatives of f.

   (b) Consider a level curve of f with z constant so $dz = 0$. Use your relation from (a) to get a general formula for the slope dy/dx of a level curve in terms of partial derivatives of f.

   (c) What condition must hold in order for you to use your result from (b) to compute dy/dx?

6. You plan to calculate the area of a long, thin rectangle from measurements of its length and width. Which dimension should you measure more carefully? Explain why?

7. Suppose that T is to be found from the formula $T = x(e^y + e^{-y})$, where x and y are found to be 2 and $ln(2)$ with maximum possible errors of $|dx| = 0.1$ and $|dy| = 0.02$ Estimate the maximum possible error in the computed value of T.