Due March 7

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Name

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use the results of discussion, a text, notes, or technology. **Only write on one side of each page.**
"*General and abstract ideas are the source of the greatest errors of mankind.*" – Rousseau, 1762

Problems

1. **You Must do this problem.**
   Let I, J be ideals of a ring R.
   (a) Show by example that $I \cup J$ need not be an ideal but show the set $I+J = \{r \in R : r = x + y,\ x \in I,\ y \in J$
   is an ideal. This ideal is called the **sum** of I and J.
   (b) Prove that $I \cap J$ is an ideal.
   (c) Show by example that the set of products $\{xy : x \in I,\ y \in J\}$ need not be an ideal but that the set of finite sums $\sum_{i,j} x_i y_j$ of products of elements of I and J is an ideal. This ideal is called the **product** ideal.
   (d) Prove $IJ \subset I \cap J$.
   (e) Show by example that IJ and $I \cap J$ need not be equal.

2. Do both of the following
   (a) For which integers n does $x^2 + x + 1$ divide $x^4 + 3x^3 + x^2 + 6x + 10$ in $(\mathbf{Z}/n\mathbf{Z})[x]$?
   (b) Describe the kernel of the map defined by $\phi : \mathbf{Z}[x] \to \mathbf{R}$ given by $\phi(f(x)) = f\left(1 + \sqrt{2}\right)$.

3. Do both of the following
   (a) i. Prove the kernel of the homomorphism $\phi : \mathbf{C}[x, y] \to \mathbf{C}[t]$ given by $\phi(f(x, y)) = f(t^2, t^3)$ is the principal ideal generated by the polynomial $y^2 - x^3$.
   ii. Describe the image of ϕ explicitly.

4. Prove that every nonzero ideal in the ring of Gauss integers contains a nonzero integer.

5. Let R be a commutative ring with identity of characteristic p where p is a prime. Prove that if a is nilpotent in R then $1 + a$ is unipotent, that is, some power of $1 + a$ is equal to 1. [An element in a ring is **nilpotent** if some positive integer power of the element equals 0.]