Mathematics 280 **Spring 2008**

Exam Key **Exam 1**

February 7, 2008

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Name

Technology used: ______________________ **Only write on one side of each page.**

- Show all of your work. Calculators may be used for numerical calculations and answer checking only.

Do any six (6) of the following

1. Sketch the graph of **one** (1) of the following polar equations. Include any tangent lines to the curve at the origin.

   (a) $r = \sin(3\theta)$

   (b) $r^2 = 4\cos(2\theta)$

2. Do **one** (1) of the following.

   (a) Find the area inside one loop of $r = \sin(3\theta)$
   Solution: One loop is swept out over the interval $0 \le \theta \le \pi/3$ so the total area is given by

$$\begin{aligned} A &= \frac{1}{2}\int_0^{\pi/3} \sin^2(3\theta)\,d\theta = \frac{1}{2}\int_0^{\pi/3} \frac{1}{2}\left[1 - \sin(6\theta)\right] d\theta \\ &= \frac{1}{4}\left[\theta + \frac{1}{6}\cos(6\theta)\right]_0^{\pi/3} = \frac{1}{4}\left[\frac{\pi}{3} - 0\right] + \frac{1}{24}\left[\cos(2\theta) - \cos(0)\right] \\ &= \frac{\pi}{12} + 0 \end{aligned}$$

   (b) Find the area inside one loop of $r^2 = 4\cos(2\theta)$
   Solution: One quarter of the graph is swept out over the interval $0 \le \theta \le \pi/4$ so the total area is given by

$$\begin{aligned} A &= 4\left(\frac{1}{2}\right)\int_0^{\pi/4} (4\cos(2\theta))\,d\theta = 8\int_0^{\pi/4} \cos(2\theta)\,d\theta \\ &= 8\left[\frac{1}{2}\sin(2\theta)\right]_0^{\pi/4} = 4\sin(\pi/2) - 4\sin(0) \\ &= 4 \end{aligned}$$

3. Use simplified equations or inequalities to describe the set of points $P(x, y, z)$ that are the same distance from the point $P_1(1, 2, 3)$ as from $P_2(-1.0, 0)$. What is your geometric intuition for the shape of this set of points?

   Solution: We are givent that the two distances $\left\|\vec{P} - \vec{P}_1\right\|$ and $\left\|\vec{P} - \vec{P}_2\right\|$ are equal. So we have

$$\begin{aligned} \left\|\vec{P} - \vec{P}_1\right\| &= \left\|\vec{P} - \vec{P}_2\right\| \\ \sqrt{(x-1)^2 + (y-2)^2 + (y-3)^2} &= \sqrt{(x+1)^2 + y^2 + z^2} \\ x^2 - 2x + 1 + y^2 - 4y + 4 + z^2 - 6z + 9 &= x^2 + 2x + 1 + y^2 + z^2 \\ -2x - 4y - 6z + 14 &= 2x + 1 \\ 4x + 4y + 6z &= 13 \end{aligned}$$

This is a plane through the midpoint of segment P_1P_2 and perpendicular to that segment.

4. Do **one** of the following.

   (a) Find the coordinates of the point Q that is 3/8 of the way along the line segment from $P_1\,(2,2,3)$ to $P_2\,(-2,5,-1)$.

   Solution: Using coordinate vectors we note that the point Q is the tip of the position vector

$$\begin{aligned}\overrightarrow{OQ} &= \overrightarrow{OP_1}+\frac{3}{8}\overrightarrow{P_1P_2}=\overrightarrow{OP_1}+\frac{3}{8}\left(\overrightarrow{OP_2}-\overrightarrow{OP_1}\right)\\ &= \langle 2,2,3\rangle+\frac{3}{8}\langle -2-2,5-2,-1-3\rangle\\ &= \langle 2,2,3\rangle+\left\langle\frac{-12}{8},\frac{9}{8},\frac{-12}{8}\right\rangle\\ &= \left\langle\frac{1}{2},\frac{25}{8},\frac{3}{2}\right\rangle\end{aligned}$$

   So Q has coordinates $\left(\frac{1}{2},\frac{25}{8},\frac{3}{2}\right)$.

   (b) Find a number c for which the angle between the vectors $\langle 1,2,1\rangle$ and $\langle 1,0,c\rangle$ equal to $\pi/3$.

   Solution: We know that the pertinent angle satisfies $\vec{a}\cdot\vec{b}=\|\vec{a}\|\left\|\vec{b}\right\|\cos(\pi/3)$ so we have the following simplifications

$$\begin{aligned}1+0+c &= \sqrt{6}\sqrt{1+0+c^2}\left(\frac{1}{2}\right)\\ (1+c)^2 &= \frac{3}{2}\left(1+c^2\right)\\ 1+2c+c^2 &= \frac{3}{2}+\frac{3}{2}c^2\\ 0 &= \frac{1}{2}c^2-2c+\frac{1}{2}\\ 0 &= c^2-4c+1\end{aligned}$$

   from which we can use the Quadratic Formula to find that

$$\begin{aligned}c &= \frac{4\pm\sqrt{16-4}}{2}\\ &= 2\pm\sqrt{3}\end{aligned}$$

5. Given $\overrightarrow{a}=<\frac{1}{\sqrt{2}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{6}}>$, and $\overrightarrow{b}=<0,\frac{1}{\sqrt{2}},-1>$ find

   (a) The scalar component (scalar projection) of $\overrightarrow{b}$ in the direction of $\overrightarrow{a}$.

   Solution: The scalar component of $\vec{b}$ in the direction of $\vec{a}$ is $\frac{\vec{b}\cdot\vec{a}}{\|\vec{a}\|}=\frac{0+\frac{1}{\sqrt{6}}-\frac{1}{\sqrt{6}}}{1}=0$

   (b) The vector projection of $\overrightarrow{b}$ in the direction of $\overrightarrow{a}$.

   Solution: The desired vector projection is $Proj_{\vec{a}}\vec{b}=\frac{\vec{b}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\vec{a}=\frac{0}{1}\vec{a}=\vec{0}$.

6. Write $\overrightarrow{b}=<8,4,-12>$ as the sum of a vector parallel to $\overrightarrow{a}=<1,2,-1>$ and a vector orthogonal to $\overrightarrow{a}$.

Solution: We saw in class that the vector $Proj_{\vec{a}}\vec{b}$ is parallel to $\vec{a}$ and that $\vec{b} - Proj_{\vec{a}}\vec{b}$ is orthogonal to $\vec{a}$. so

$$\begin{aligned}
\vec{b} &= Proj_{\vec{a}}\vec{b} + \left(\vec{b} - Proj_{\vec{a}}\vec{b}\right) \\
&= \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\vec{a} + \left(\vec{b} - \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\vec{a}\right) \\
&= \frac{8+8+12}{1+4+1}\vec{a} + \left(\vec{b} - \frac{8+8+12}{1+4+1}\vec{a}\right) \\
&= \frac{14}{3}\langle 1,2,-1\rangle + \left(\langle 8,4,-12\rangle - \frac{14}{3}\langle 1,2,-1\rangle\right) \\
&= \left\langle \frac{14}{3}, \frac{28}{3}, \frac{-14}{3}\right\rangle + \left\langle \frac{24-14}{3}, \frac{12-28}{3}, \frac{-36+14}{3}\right\rangle \\
&= \left\langle \frac{14}{3}, \frac{28}{3}, \frac{-14}{3}\right\rangle + \left\langle \frac{10}{3}, \frac{-16}{3}, \frac{-22}{3}\right\rangle
\end{aligned}$$

So the desired vectors are $\left\langle \frac{14}{3}, \frac{28}{3}, \frac{-14}{3}\right\rangle$ and $\left\langle \frac{10}{3}, \frac{-16}{3}, \frac{-22}{3}\right\rangle$

7. Find the angle between the diagonal of a cube and one of the edges the diagonal meets at a vertex.

   Solution: If you draw a cube of side length c in the first octant with one corner at the origin and all faces parallel to coordinate planes, then the vector from the origin to $\langle c,c,c\rangle$ is a diagonal of the cube and the vector from the origing to $\langle c,0,0\rangle$ is an edge of the cube with the same base point as the diagonal. Thus, the angle, θ, between these two vectors can be computed using the dot product:

$$\begin{aligned}
\cos(\theta) &= \frac{\langle c,c,c\rangle\cdot\langle c,0,0\rangle}{\|\langle c,c,c\rangle\|\,\|\langle c,0,0\rangle\|} \\
&= \frac{c^2}{\sqrt{3c^2}\sqrt{c^2}} = \frac{c^2}{\sqrt{3}c^2} \\
&= \frac{1}{\sqrt{3}}
\end{aligned}$$

   so $\theta = \arccos\left(\frac{1}{\sqrt{3}}\right)$.

8. Given vectors $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$, use the dot product to write formulas for the following.

   (a) The vector projection of $\overrightarrow{a}$ onto $\overrightarrow{b}$.

   Solution: $Proj_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\vec{b}$

   (b) A vector with the length of $\overrightarrow{a}$ and the direction of $\overrightarrow{b}$.

   Solution: This is the vector $\vec{c} = \|\vec{a}\|\left(\frac{\vec{b}}{\|b\|}\right) = \left(\frac{\sqrt{\vec{a}\cdot\vec{a}}}{\sqrt{\vec{b}\cdot\vec{b}}}\right)\vec{b}$