Mathematics 280 **Spring 2008**

Exam 2

February 28, 2008

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Name

Technology used: ______________________

Only write on one side of each page.

- Show all of your work. Calculators may be used for numerical calculations and answer checking only.

Exam Breakdown by Problem Type

- 65 Computation points – all selectable as problems – one directly from assigned homework
- 15 'synthesize two Computation problems' points – 9 of which are purely Computation points
- 45 One Step points – all selectable as problems

Do BOTH of these problems

A.1 (10 points) If $\overrightarrow{a}$, $\overrightarrow{b}$,and $\overrightarrow{c}$ are vectors in $\mathbf{R}^3$, state whether each expression is meaningful. If it is, state whether it is a scalar or a vector.

(a) $\overrightarrow{a} \cdot \left(\overrightarrow{b} \times \overrightarrow{c}\right)$ (scalar)

(b) $\overrightarrow{a} \times \left(\overrightarrow{b} \cdot \overrightarrow{c}\right)$ (makes no sense)

(c) $\overrightarrow{a} \times \left(\overrightarrow{b} \times \overrightarrow{c}\right)$ (vector)

(d) $\left(\overrightarrow{a} \cdot \overrightarrow{b}\right) \times \overrightarrow{c}$ ((makes no sense)

(e) $\left(\overrightarrow{a} \cdot \overrightarrow{b}\right) \times \left(\overrightarrow{c} \cdot \overrightarrow{d}\right)$ (makes no sense)

(f) $\left(\overrightarrow{a} \times \overrightarrow{b}\right) \cdot \left(\overrightarrow{c} \times \overrightarrow{d}\right)$(scalar)

(g) $\left[\left(\vec{a} \times \vec{b}\right) \cdot \vec{c}\right] \vec{c} \cdot \vec{a}$ (scalar)

Solution: (Similar to Computation problem #31 on page 642.**)**
The cross product outputs a vector and the dot product outputs a scalar. Neither a dot product nor cross product makes sense unless both factors are vectors.

A.2 (10 points) Convert **three** (3) of these equations to rectangular coordinates.

(a) $z = r^2 \sin(2\theta) = 2r\sin(\theta)\, r\cos(\theta) = 2xy$

(b) $r = \sin(\theta)$ implies $r^2 = r\sin(\theta)$ so $x^2 + y^2 = y$

(c) $\rho^2 \sin^2(\phi) = 1$ implies $r^2 = 1$ so $x^2 + y^2 = 1$

(d) $\rho^2 \sin(\phi)\cos(\phi)\cos(\theta) = 1$ implies $[\rho\sin(\phi)\cos(\theta)](\rho\cos(\phi)) = 1$ so $xz = 1$

Solution: (Assigned Computation homework problem for section 13.7**)**

Do any two (2) of the following

B.1 Express the tangent line to the curve $\overrightarrow{F}(t) = \langle t^2, e^{t-1}, \ln(t)\rangle$ at the point where $t = 1$ in parametric form.

Solution: (Similar to Computation problems 19, 21 **on page** 670**)**
A point on the tangent line is given by $\overrightarrow{F}(1) = \langle 1^2, e^0, \ln(1)\rangle = \langle 1, 1, 0\rangle$. The tangent line is in the direction of $\overrightarrow{F}'(1) = \langle 2, 1, 1\rangle$ since $\overrightarrow{F}'(t) = \left\langle 2t, e^{t-1}, \frac{1}{t}\right\rangle$.So the tangent line has parametrized form: $x = 1 + 2t$, $y = 1 + t$, $z = 0 + t$.

B.2 Write an equation for the plane that passes through the point $P(6, 0, -2)$ and contains the line $x = t,\ y = \frac{1}{2}t,\ z = \frac{1}{3}t$.

Solution: (One Step beyond assigned Computation problem 23 **on page** 650**)**
The direction vector of the line $\vec{d} = \langle 1, 1/2, 1/3\rangle$ is in the plane as is the origin $O(0, 0, 0)$ since that is the point given on the line by $t = 0$. Thus $\overrightarrow{OP} = \langle 6, 0, -2\rangle$ is also on the plane. Hence, a normal vector is $\vec{n} = \vec{d} \times \overrightarrow{OP} = \langle 1, 1/2, 1/3\rangle \times \langle 6, 0, -2\rangle = \langle -1, 4, -3\rangle$. Thus an equation for the plane is: $(-1)(x-6) + 4(y-0) - 3(z+2) = 0$ which simplifies to $x - 4y + 3z = 0$.

B.3 What is the distance (measured along a line orthogonal to both) between the parallel planes $x + 2y - 3z = 1$ and $x + 2y - 3z = 50$? [The answer is not 49.]

Solution: (Similar to One Step problem 45 **on page** 650. **Also, one step beyond Computation problem** 41**on page** 650**)**
A point on the first plane is $P(1, 0, 0)$ and a point on the second is $Q(50, 0, 0)$. Thus $\overrightarrow{PQ} = \langle 49, 0, 0\rangle$ is a vector running from one plane to the other. Thus, the distance between the two planes is the magnitude of the vector projection of $\overrightarrow{PQ}$ onto the normal vector $\vec{n}$. Hence the distance is the magnitude of

$$\begin{aligned}
\mathrm{Proj}_{\vec{n}}\overrightarrow{PQ} &= \frac{\overrightarrow{PQ}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\vec{n} \\
&= \frac{\langle 49, 0, 0\rangle \cdot \langle 1, 2, -3\rangle}{\langle 1, 2, -3\rangle \cdot \langle 1, 2, -3\rangle} \cdot \langle 1, 2, -3\rangle \\
&= \frac{49}{14}\langle 1, 2, -3\rangle
\end{aligned}$$

Finally, the distance we seek is the magnitude of this last vector: $\frac{49}{14}\sqrt{14} = \frac{7}{2}\sqrt{7}\sqrt{2} = \frac{7\sqrt{7}}{\sqrt{2}}$.

Do any three (3) of the following

C.1 Identify **three** of the following quadric surfaces by name and quickly sketch **one** of them.

(a) $y^2 + z^2 = 1 - 4x^2$: Ellipsoid

(b) $y^2 + z^2 = x$: Paraboloid opening in the positive x direction

(c) $y^2 + z^2 = 1$: cylinder with circular cross sections opening in the x direction

(d) $x^2 + z^2 = 1 + y^2$: Hyperboloid of one sheet opening in the y direction.

Solution: (Similar to Computation exercises $1 - 12$ and $13 - 44$ on page 656)

C.2 Suppose $\overrightarrow{r}''(t) = (3t^2 + 4)\,\mathbf{i} + \mathbf{j} + 4t\,\mathbf{k}$. If $\overrightarrow{r}(0) = \mathbf{i} + \mathbf{j}$, and $\overrightarrow{r}'(0) = \overrightarrow{0}$ find the function $\overrightarrow{r}(t)$.

Solution: (Similar to Computation problems $9, 11, 13$ on page 676)
Integrating we have $\vec{r}'(t) = \langle t^3 + 4t, t, 2t^2 \rangle + \vec{C}_1$ and the initial condition $\overrightarrow{r}'(0) = \overrightarrow{0}$ tells us that $\vec{C}_1 = \vec{0}$. Integrating again we have $\vec{r}(t) = \left\langle \frac{1}{4}t^4 + 2t^2, \frac{1}{2}t^2, \frac{2}{3}t^3 \right\rangle + \vec{C}_2$ and the initial condition tells us that $\vec{C}_2 = \langle 1, 1, 0 \rangle$ so that the final solution is $\vec{r}(t) = \left\langle \frac{1}{4}t^4 + 2t^2 + 1, \frac{1}{2}t^2 + 1, \frac{2}{3}t^3 \right\rangle$

C.3 Find the point of intersection of $\overrightarrow{r}_1(t) =< t, 1 - t, 3 + t^2 >$ and $\overrightarrow{r}_2(s) =< 3 - s, s - 2, s^2 >$ and compute the angle between the tangent vectors at this point.

Solution: (Intersection part is similar to Computation problem 61 and Turn In Problems $73, 74$ on pages $650, 651$. Tangent vectors similar to Computation problems $19, 21$ on page 670. Angle between vectors portion from section 10.3)
Any point of intersection must satisfy the system of equations

$$\left\{ \begin{array}{c} t = 3 - s \\ 1 - t = s - 2 \\ 3 + t^2 = s^2 \end{array} \right\} \rightarrow \left\{ \begin{array}{c} t = 3 - s \\ 1 - (3 - s) = s - 2 \\ 3 + (9 - 6s + s^2) = s^2 \end{array} \right\}$$

which we can easily see is satisfied if and only if $s = 2$ and $t = 1$. Thus the point of intersection is at the tip of the position vector $\overrightarrow{r}_1(1) =< 1, 0, 4 >= \overrightarrow{r}_2(2)$.
The tangent vectors at this point are $\vec{r}_1'(1) =< 1, -1, 2 >$ and $\vec{r}_2'(2) =< -1, 1, 4 >$. The angle between these two vectors is $\arccos\left(\frac{<1,-1,2>\cdot<-1,1,4>}{\|<1,-1,2>\|\|<-1,1,4>\|}\right) = \arccos\left(\frac{6}{\sqrt{6}\sqrt{18}}\right) = \arccos\left(\frac{1}{\sqrt{3}}\right)$

C.4 The paraboloid $2y = (x-1)^2 + z^2$ and the plane $x + z = 1$ intersect along a curve in $\mathbf{R}^3$. Find a parametrization $\overrightarrow{F}(t)$ for this curve.

Solution: (A One Step question similar to examples done in class and the study session.)
Rewriting the equation of the plane we see that $x - 1 = -z$. Thus, any point (x, y, z) that satisfies both equations must also satisfy

$$\begin{aligned} 2y &= (x-1)^2 + z^2 \\ &= (-z)^2 + z^2 \end{aligned}$$

so that $y = z^2$. We thus see that $y = z^2$ and $x = 1 - z$ so we can parametrize the intersection of the two surfaces by setting $z = t$ obtaining

$$\vec{r}(t) = \left\langle 1 - t, t^2, t \right\rangle$$

or in parametrized form: $x = 1 - t$, $y = t^2$, $z = t$.