Turn In 4.2

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Problem Statement

Here is another typical problem in computer graphics. Your eye is at $E(4,0,0)$. You are looking at a triangular plate whose vertices are at $P(1,0,1)$, $Q(1,1,0)$, and $R(-2,2,2)$. The line segment from $A(1,0,0)$ to $B(0,2,2)$ passes through the plate. What portion of the line segment is hidden from your view by the plate. [This is an exercise in finding the intersections of lines and planes.]

Comments

Many people misinterpreted the problem in that they thought the line segment started out in front of the plate and passed behind it at some point. Their solutions involved finding the point of intersection of the line segment AB and the plane and then describing the portion of the segment with smaller x coordinates (further from the eye).

Many other people saw that the beginning of the segment and the beginning of the plate both occur in the plane $x = 1$ with the segment starting at a lower point (smaller z value). Hence they realized that the segment starts below the plate and passes through it from bottom to top making the part of the segment furthest from the eye easily seen since it is not obscured by the plate. These people also realized that the part of the segment closest to the point A is also in view and so they needed to figure out at what point the segment passes behind (from the point of view of the eye) the plate as well as the point where it passes through the plate and becomes visible again. I looked up the answer in the instructors solution key and the person who solved the problem there did not understand this last part of the problem. They just computed the part of the line segment starting from A and ending at the point where the line segment intersected the plate.

Because the eye is at $E(4,0,0)$ and the segment and plate start in the plane $x = 1$, there is some subtlety in determining at which point the segment first passes behind the plate. This is the point X on the segment AB that lies on the line passing through segment PQ (the start of the plate) and point E (the eye).

Solution – part 1

- First we write the parametrization of the line segment AB. We use the point $A(1,0,0)$ and direction vector $\vec{d} = \langle -1, 2, 2\rangle$ to obtain $\vec{r}(t) = \langle 1-t, 2t, 2t\rangle$, $0 \le t \le 1$.

- Second we write the parametrization of the line segment PQ using point $P(1,0,1)$ and direction vector $\vec{c} = \langle 0, 1, -1\rangle$ obtaining $\vec{g}(u) = \langle 1, u, 1-u\rangle$, $0 \le u \le 1$.

- Now, for each t in $0 \le t \le 1$ we parametrize the line L_t from the eye $E(4,0,0)$ to the point at the tip of the position vector $\vec{r}(t) = \langle 1-t, 2t, 2t\rangle$. Our first goal is to find the point where this line, L_t intersects segment PQ (at point X). Our second (and primary) goal is to then find the point, Z, on L_t that is also on segment AB. This is the first point on segment AB that is obscured from the view of the eye since Z lies directly behind (from the viewpoint of the eye) segment AB which is the base of the plate.

- Remember, it is the point Z that we are looking for. This is the first point on segment AB that we cannot see from our eye location at E.

Finding L_t

- For this line, we use the point $E\left(4,0,0\right)$ and the direction vector from $E\left(4,0,0\right)$ to the point at the tip of $\vec{r}\left(t\right)=\langle 1-t,2t,2t\rangle$. To avoid ambiguity, we use the letter s as the parameter of this line, L_t.

- The direction vector is $\vec{D}=\langle 1-t-4,2t-0,2t-0\rangle=\langle -3-t,2t,2t\rangle$ so the parametrized form for the line is $\vec{L}_t\left(s\right)=\langle 4,0,0\rangle+s\ \langle -3-t,2t,2t\rangle=\langle 4-3s-st,2st,2st\rangle$.

- Any point that lies on both $\vec{L}_t\left(s\right)$ and segment PQ must correspond to values of s and u so that $\vec{L}_t\left(s\right)=\vec{g}\left(u\right)$ which means that $\langle 4-3s-st,2st,2st\rangle=\langle 1,u,1-u\rangle$. This gives rise to the system of equations
$$\begin{cases} 4-3s-st=1 \\ 2st=u \\ 2st=1-u \end{cases}$$
Using the last two equations it is easy to see that $u=1-u$ so that $u=1/2$. And using the middle equation with $u=1/2$ we see that $s=\frac{1}{4t}$. If we now substitute this value of s into the top equation we get
$$\begin{aligned} 4-3\left(\frac{1}{4t}\right)-\frac{t}{4t} &= 1 \\ \frac{16t-3-t}{4t} &= 1 \\ 15t-3 &= 4t \\ t &= \frac{3}{11} \end{aligned}$$

- Plugging this value of t into the parametrization $\vec{r}\left(t\right)$ for line segment AB we get $\vec{r}\left(3/11\right)=\langle 1-3/11,2\left(3/11\right),2\left(3/11\right)\rangle=\frac{1}{11}\langle 8,6,6\rangle$. And plugging $u=\frac{1}{2}$ into our parametrization for segment PQ we get $\vec{g}\left(\frac{1}{2}\right)=\left\langle 1,\frac{1}{2},\frac{1}{2}\right\rangle$.

- If we have done this correctly then the points $E\left(4,0,0\right)$, $Z\left(\frac{8}{11},\frac{6}{11},\frac{6}{11}\right)$, and $X\left(1,\frac{1}{2},\frac{1}{2}\right)$ should be colinear. We check this by showing that all three are on the line $\vec{L}_t\left(s\right)$ where $t=\frac{3}{11}$. That is the line with parametrization $\left\langle 4-3s-s\frac{3}{11},2s\left(\frac{3}{11}\right),2s\left(\frac{3}{11}\right)\right\rangle$
$$\begin{aligned} \vec{L}_{3/11}\left(s\right) &= \left\langle 4-3s-s\frac{3}{11},2s\left(\frac{3}{11}\right),2s\left(\frac{3}{11}\right)\right\rangle \\ &= \left\langle 4-3s-\frac{3s}{11},\frac{6s}{11},\frac{6s}{11}\right\rangle \\ &= \left\langle \frac{44-36s}{11},\frac{6s}{11},\frac{6s}{11}\right\rangle \end{aligned}$$

- Note that

1. $\vec{L}_{3/11}(0) = \langle 4, 0, 0\rangle$ so E is on this line.
2. $\vec{L}_{3/11}(1) = \left\langle \frac{44-36}{11}, \frac{6}{11}, \frac{6}{11}\right\rangle = \left\langle \frac{8}{11}, \frac{6}{11}, \frac{6}{11}\right\rangle$ so Z is on this line and finally
3. $\vec{L}_{3/11}\left(\frac{11}{12}\right) = \left\langle \frac{44-36(11/12)}{11}, \frac{6(11/12)}{11}, \frac{6(11/12)}{11}\right\rangle = \left\langle \frac{44-33}{11}, \frac{1}{2}, \frac{1}{2}\right\rangle = \left\langle 1, \frac{1}{2}, \frac{1}{2}\right\rangle$ so X is on the line.

Solution – Part 2

It is straightforward to determine a normal vector to the plane containing P, Q, and R. One such normal is $\langle 3, 3, 3\rangle$. An equation for this plane is $x + y + z = 2$. The point where this plane meets the line segment AB is found by substituting the components of the parametrization for the line segment $\vec{r}(t) = \langle 1 - t, 2t, 2t\rangle$ into the equation of the plane and solving for $t = \frac{1}{3}$. The resulting point is $Y\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$.
Thus the solution to the problem as stated in the text is: the portion of the line segment obscured from view is the portion between $\vec{r}\left(\frac{3}{11}\right)$ and $\vec{r}\left(\frac{1}{3}\right)$. This turns out to be $\frac{2}{33}$ of the entire segment.