Mathematics 181D **Spring 2008**

Exam Key **Exam 1**

February 7, 2008

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Name

Technology used: ______________________________________ **Only write on one side of each page.**

- Show all of your work. Calculators may be used for numerical calculations and answer checking only.

Do both of these problems

1. Find the derivative of $F(x) = \int_x^{e^x} t^2\ dt$

   (a) Using part 1 of the Fundamental Theorem of Calculus
   Solution: $F(x) = \int_x^0 t^2 dt + \int_0^{e^x} t^2 dt = -\int_0^x t^2 dt + \int_0^{e^x} t^2 dt$.Now applying the Fundamental Theorem of Calculus and the Chain Rule we get

$$\begin{aligned} F'(x) &= -x^2 + (e^x)^2 \frac{d}{dx}[e^e] \\ &= -x^2 + e^{3x} \end{aligned}$$

   (b) Using part 2 of the Fundamental Theorem of Calculus
   Solution: Since $\int t^2 dt = \frac{1}{3}t^3 + C$ then by the Fundamental Theorem of Calculus

$$\begin{aligned} F(x) &= \int_x^{e^x} t^2\ dt = \frac{1}{3}t^3\Big]_x^{e^x} \\ &= \frac{1}{3}(e^x)^3 - \frac{1}{3}x^3 \\ &= \frac{1}{3}e^{3x} - \frac{1}{3}x^3 \end{aligned}$$

   Which tells us that

$$\begin{aligned} F'(x) &= 3\left(\frac{1}{3}e^{3x}\right) - x^2 \\ &= e^{3x} - x^2 \end{aligned}$$

2. Use substitution to evaluate two of the following indefinite integrals.

   (a) $\int \frac{1}{t^3}\sin\left(3 - \frac{1}{t^2}\right) dt$
   Solution: Using the rule of thumb substitution $u = 3 - \frac{1}{t^2}$ we compute $du = \left(0 + \frac{2}{t^3}\right) dt$ so that

$$\begin{aligned} \int \frac{1}{t^3}\sin\left(3 - \frac{1}{t^2}\right) dt &= \frac{1}{2}\int \sin(u)\, du = -\frac{1}{2}\cos(u) + C \\ &= -\frac{1}{2}\cos\left(3 - \frac{1}{t^2}\right) + C \end{aligned}$$

(b) $\int \frac{\sqrt{3+\arctan(x)}}{1+x^2}dx$

Solution: We use the rule of thumb substitution $u = 3 + \arctan(x)$ which gives $du = \frac{1}{1+x^2}dx$ so that

$$\begin{aligned}
\int \frac{\sqrt{3+\arctan(x)}}{1+x^2}dx &= \int u^{1/2}du \\
&= \frac{u^{3/2}}{3/2} + C \\
&= \frac{2}{3}(3+\arctan(x))^{3/2} + C
\end{aligned}$$

(c) $\int \sqrt{1-x^2}dx$ start by using the substitution $x = \sin(\theta)$.

Solution: Since $x = \sin(\theta)$ we know that $dx = \cos(\theta)\,d\theta$ and that $\sqrt{1-x^2} = \sqrt{1-\sin^2(\theta)} = \sqrt{\cos^2(\theta)} = \cos(\theta)$. We also know that $x = \sin(\theta)$ implies that $\theta = \arcsin(x)$ .By not forgetting to keep track of the " dx" part, we get

$$\begin{aligned}
\int \sqrt{1-x^2}dx &= \int \cos(\theta)\cos(\theta)\,d\theta \\
&= \int \cos^2(\theta)\,d\theta \\
&= \frac{1}{2}\int (1+\cos(2\theta))\,d\theta \\
&= \frac{1}{2}\int 1d\theta + \frac{1}{2}\int \cos(2\theta)\,d\theta
\end{aligned}$$

Now making the substitution $u = 2\theta$ in the second integral we get $du = 2d\theta$ yielding

$$\begin{aligned}
\int \sqrt{1-x^2}dx &= \frac{1}{2}\int 1d\theta + \frac{1}{4}\int \cos(u)\,du \\
&= \frac{1}{2}\theta + \frac{1}{4}\sin(u) + C \\
&= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C \\
&= \frac{1}{2}\theta + \frac{1}{4}(2\sin(\theta)\cos(\theta)) + C \\
&= \frac{1}{2}\arcsin(x) + \frac{1}{2}x\sqrt{1-x^2} + C
\end{aligned}$$

Do any four (4) of the following problems

1. Solve the initial value problem

$$\frac{d^3y}{dx^3} = x, \quad \left.\frac{d^2y}{dx^2}\right|_{x=0} = 2, \quad y'(0) = 3, \quad y(0) = 1$$

Solution: $\frac{d^3y}{dx^3} = x$ tells us that $y'' = \frac{1}{2}x^2 + C_1$. Since $y''(0) = 2$ we can see that $C_1 = 2$ and $y'' = \frac{1}{2}x^2 + 2$. This in turn tells us that $y' = \frac{1}{6}x^3 + 2x + C_2$ and $y'(0) = 3$ tells us that $C_2 = 3$ so $y' = \frac{1}{6}x^3 + 2x + 3$. Now y is the antiderivative of this function so it must be $y = \frac{1}{24}x^4 + x^2 + 3x + C_3$ and the last initial condition of $y(0) = 1$ implies $c_3 = 1$ so the final solution is that $y(x) = \frac{1}{24}x^4 + x^2 + 3x + 1$.

2. Without using a calculator, evaluate **both** of the following indefinite integrals

(a) $\int \left(7\sec^2(x) - \frac{2}{1+x^2} + \sec(x)\tan(x) + \frac{1}{x^{3/4}}\right)\ dx$

Solution: These are all functions whose antiderivatives are known formulas although we need to write $\frac{1}{x^{3/4}}$as $x^{-3/4}$ in order to apply the power rule. The anser is

$$7\tan(x) - 2\arctan(x) + \sec(x) + \frac{x^{1/4}}{1/4} + C$$

(b) $\int \frac{1}{y^2}\left(2y^3 + 3y^2 + 4y + y^{1/2}\right)\ dy$

Solution: First we multiply through by $\frac{1}{y^2}$and then apply know rules.

$$\begin{aligned} \int \frac{1}{y^2}\left(2y^3 + 3y^2 + 4y + y^{1/2}\right)\ dy &= \int \left(2y + 3 + \frac{4}{y} + y^{-3/2}\right)\ dy \\ &= y^2 + 3y + 4\ln[y] + \frac{y^{-1/2}}{-1/2} + C \end{aligned}$$

3. If we use the partition points $x_0 < x_1 < x_2 < \cdots < x_n$ to partition the interval $[1,5]$ into n subintervals of equal length.

(a) What is the value of Δx in terms of the letter n? **Solution:** $\Delta x = \frac{5-1}{n} = \frac{4}{n}$

(b) Write the values of x_0, x_1, x_2, x_k, and x_n in terms of the letter n. **Solution:** $x_0 = 1 + 0\frac{4}{n}1$, $x_1 = 1 + \frac{4}{n}$, $x_2 = 1 + 2\frac{4}{n}$, $x_k = 1 + k\frac{4}{n}$, $x_n = 1 + n\frac{4}{n} = 5$

(c) Use sigma notation to write, in terms of the letter n, the Riemann sum for the function $f(x) = x - x^2$ that uses the left endpoint of each subinterval as the value of c_k.**Do not simplify this Riemann Sum.**

Solution: Either of the following is correct.

$$\sum_{k=0}^{n-1}\left[\left(1 + k\frac{4}{n}\right) - \left(1 + k\frac{4}{n}\right)^2\right]\frac{4}{n} = \sum_{k=1}^{n}\left[\left(1 + (k-1)\frac{4}{n}\right) - \left(1 + (k-1)\frac{4}{n}\right)^2\right]\frac{4}{n}$$

4. If we partition the interval $[0,3]$ into n subintervals of equal width, then the Riemann sum for the function $f(x) = 4x - x^3$ that uses this partition and the right endpoint of each subinterval as the value of c_k is $\sum_{k=1}^{n}\left[4\left(0 + \frac{3k}{n}\right) - \left(0 + \frac{3k}{n}\right)^3\right]\frac{3}{n}$.

(a) Use limits to compute the value of $\int_0^3 (4x - x^3)\ dx$. [No credit if you use the Fundamental Theorem of Calculus.]

Useful facts: $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$, $\sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n+1)^2$.

Solution:

$$\begin{aligned} \sum_{k=1}^{n}\left[4\left(0 + \frac{3k}{n}\right) - \left(0 + \frac{3k}{n}\right)^3\right]\frac{3}{n} &= \sum_{k=1}^{n}\left[4\left(\frac{3k}{n}\right) - \left(\frac{3k}{n}\right)^3\right]\frac{3}{n} \\ &= \sum_{k=1}^{n}\left[\frac{12k}{n} - \frac{27k^3}{n^3}\right]\frac{3}{n} \\ &= \sum_{k=1}^{n}\left[\frac{36k}{n^2} - \frac{81k^3}{n^4}\right] \\ &= \frac{36}{n^2}\sum_{k=1}^{n} k - \frac{81}{n^4}\sum_{k=1}^{n} k^3 \\ &= \frac{36}{n^2}\frac{n(n+1)}{2} - \frac{81}{n^4}\frac{n^2(n+1)^2}{4} \\ &= 18\left(1 + \frac{1}{n}\right) - \frac{81}{4}\left(1 + \frac{1}{n}\right)^2 \end{aligned}$$

Now taking the limit as $n \to \infty$ of this last expression we see that $\int_0^3 (4x - x^3) = \lim_{n\to\infty} \left[18\left(1+\frac{1}{n}\right) - \frac{81}{4}\left(1+\frac{1}{n}\right)^2\right] = 18 - \frac{81}{4} = -\frac{9}{4}$

5. Given the function $f(x) = \sqrt{x^2+1}$ with domain the interval $[0,5]$. Write a Riemann sum for f using a partition P that divides $[0,5]$ into 3 subintervals and where $\|P\| = 2$. Be sure to specify the partition points of P as well as writing out the Riemann Sum **without** using sigma notation.

   Solution: There are many possible partitions. One that guarantees that $\|P\| = 2$ is to use $x_0 = 0$, $x_1 = 2$, $x_2 = 4$, and $x_5 = 5$. Note that $\Delta x_1 = 2$, $\Delta x_2 = 2$ and $\Delta x_3 = 1$. Then the Riemann sum that uses the left endpoints of each subinterval as c_k is

$$2\sqrt{0^2+1} + 2\sqrt{2^2+1} + (1)\sqrt{4^2+1}$$

6. Suppose that f and g are integrable functions and that $\int_a^b (3f(x) - g(x))\, dx = 5$ and $\int_a^b (f(x) + g(x))\, dx = 7$. Use properties of definite integrals to find $\int_a^b f(x)\, dx$ and $\int_a^b g(x)\, dx$.

   Solution: For the moment, let $A = \int_a^b f(x)\, dx$ and $B = \int_a^b g(x)\, dx$. Then the first of our two equations can be rewritten as $\int_a^b (3f(x) - g(x))\, dx = \int_a^b 3f(x)\, dx - \int_a^b g(x)\, dx = 3\int_a^b f(x)\, dx - \int_a^b g(x)\, dx = 3A - B = 5$ and the other one as $A + B = 7$ solving this second equation for $B = 7 - A$ and substituting back into the first equation we get $3A - (7 - A) = 5$ which simplifes to $4A = 12$ so $A = 3$. Putting this back into $B = 7 - A$ we see that $B = 4$. Thus the solution is

$$\begin{aligned} \int_a^b f(x)\, dx &= A = 3 \\ \int_a^b g(x)\, dx &= B = 4 \end{aligned}$$