Mathematics 181 **Spring 2007** **Quiz 7**

Due April 13, 2007

Name

Only write on one side of each page.
Be sure to re-read the WRITING GUIDELINES rubric, since it defines how your project will be graded. In particular, you may discuss this project with others but **you may not collaborate on the written exposition of the solution.**
"It is hard to know what you are talking about in mathematics, yet no one questions the validity of what you say. There is no other realm of discourse half so queer." – J. R. Newman

The Background

In many ways, the 1600 and 1700's were the period when the ideas that underpin what we now call calculus were being developed. During this period there was a rich correspondence among those we would now call mathematicians, physicists or chemists. Often, one person would pose a question that he (or very rarely, she) had been studying and others would try to solve it. In some ways it was a kind of competition.
One such problem (which we might address later in the semester) was posed in 1644 by Pietro Mengoli (1625-1686). It was called the Basel problem and is easily stated: What is the sum of the reciprocals of the perfect squares? Today we would say: the following series converges by the integral test; to what sum does it converge?

$$\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots$$

For nearly 100 years the best minds were not able to answer this question. However, around 1729, Leonard Euler found the sum. In fact, he found three different ways to compute it. One of those methods came from a very clever method for interpolating the outputs of functions that had integer domains. One of the functions he interpolated was the factorial function. What I mean by interpolation is this: factorials are defined only for nonnegative integers: $0! = 1,\ 1! = 1,\ 2! = 2\cdot 1,\ 3! = 3\cdot 2\cdot 1$, etc. What Euler did was to find a function with domain the set of all real numbers except the nonnegative integers whose graph passes through all of the dots that are the graph of the factorial sequence $\{n!\}$. Here is a graph of a portion of this function.

Euler defined the output of his function given input m (which need not be an integer) as

$$\frac{1\cdot 2^m}{1+m}\cdot\frac{2^{-m}\cdot 3^m}{2+m}\cdot\frac{3^{-m}\cdot 4^m}{3+m}\cdot\cdots\cdot \tag{1}$$

Since we have not studied infinite products we will not use this form of Euler's function. Instead we will limit ourselves to the comment that the logarithm of this product would be an infinite series and perhaps what we are learning about series would allow us to fully understand what Euler was doing. Instead we will work with the modern formulation of Euler's function. It is called the **Gamma function** and is extremely useful in various applications in physics, statistics, and most places where factorials tend to crop up. As I noted above, this improper integral converges for every real number except the nonnegative integers and it's graph passes through the points $(n+1, n!)$ for every nonnegative integer n.

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\ dt.$$

Evaluating this function at different values of x is not trivial. For example, computing $\Gamma(2)$ is best done using integration by parts. As a start on the process, note that

$$\begin{aligned}\Gamma(2) &= \int_0^\infty t^{2-1} e^{-t}\ dt \\ &= \int_0^\infty t e^{-t}\ dt.\end{aligned}$$

The purpose of this project is for you to explore some of the basic properties of the Gamma function using the integration tools we have developed.

The Project

1. As we explore the Gamma function, one of the tools we will need is the fact that if a is a positive constant then
$$\lim_{x\to\infty} \frac{x^a}{e^x} = 0.$$
Prove this result by
   (a) Showing $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$ for any positive integer n and
   (b) Using the sandwich theorem and part a.to show $\lim_{x\to\infty} \frac{x^a}{e^x} = 0$ for any positive constant a.

2. For this problem, you may use, without proving it, the fact that the improper integral $\int_0^\infty e^{-x^2}\,dx$ converges to the value $\sqrt{\pi}/2$. (This is usually shown in multivariable calculus and was known to Euler back in 1730.)
   (a) Directly evaluate the improper integrals for $\Gamma(1)$ and $\Gamma(2)$ using integration by parts.
   (b) Use the substitution $t = u^2$ to evaluate the improper integral for $\Gamma(1/2)$.
   (c) Use integration by parts to show that for any $x > 0$,
   $$\Gamma(x+1) = x\Gamma(x) \tag{2.}$$
      i. Using your previously computed values, check that this formula gives you the proper value of $\Gamma(2)$.
      ii. Why can't we use $\Gamma(1)$ and this formula to tell us $\Gamma(0)$? Does your proof using integration by parts work for any other values of x besides when $x > 0$?
   (d) **Without evaluating any more improper integrals,** use parts a., b., and c. to compute $\Gamma(2)$, $\Gamma(3)$, $\Gamma(3/2)$, $\Gamma(5/2)$, $\Gamma(-1/2)$, and $\Gamma(-3/2)$.
   (e) Explain why $\Gamma(n+1) = n!$ for every nonnegative integer n.
   (f) Use display 2 to explain why we can intuitively think of "$\left(\frac{1}{2}\right)!$"$= \Gamma\left(\frac{3}{2}\right)$ as the infinite product $\frac{2}{3}\cdot\frac{2}{5}\cdot\frac{2}{7}\cdot\frac{2}{9}\cdot\cdots$ and show that this agrees with Euler's formula in Display 1 when $m = \frac{1}{2}$.