## Mathematics 232-A Exam 2 Spring 2006

**February 23, 2006**

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*Name*

**Technology used:** ______________________

- **Only write on one side of each page.**
- **Use terminology correctly.**
- **Partial credit is awarded for correct approaches so justify your steps.**

### Do any three (3) of these computational problems

C.1. Is the set of vectors $S = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\} \left\{ \begin{bmatrix} 7 \\ 3 \\ 5 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 3 \\ 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 3 \\ 8 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 7 \\ 3 \\ 1 \end{bmatrix} \right\}$ linearly dependent or linearly independent? If it is linearly dependent, first write one of the $\mathbf{w}$'s as a linear combination of the others and then write the set $T$ that is a subset of $S$, is linearly independent, and for which $< T > = < S >$.

1. The set $S$ is linearly dependent because: the coefficient matrix for the system of equations $\alpha_1\mathbf{w}_1 + \alpha_2\mathbf{w}_2 + \alpha_3\mathbf{w}_3 + \alpha_4\mathbf{w}_4 = \mathbf{0}$ is $\begin{bmatrix} 7 & 2 & 4 & 4 \\ 3 & 5 & 3 & 9 \\ 5 & 3 & 3 & 7 \\ 4 & 9 & 8 & 3 \\ 2 & 7 & 6 & 1 \end{bmatrix}$ which has reduced row echelon form: $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
Since there is no leading one in the last column there are infinitely many solutions to the system and there exists a non-trivial relation of dependence for $S$.

C.2. Write each of the following complex numbers in the form $a + bi$.

(a) $i(3-2i) + 7\left(\overline{-2+i}\right) = (3i+2) + 7(-2-i) = -12 - 4i$.

(b) $(4-2i)(-3+i) = -10 + 10i$

(c) $\frac{2-i}{3+4i} = \frac{2}{25} - \frac{11}{25}i$ [Multiply by $\frac{3-4i}{3-4i}$and simplify.]

C.3. Consider the following vectors in $\mathbf{C}^4$.

$$\vec{v}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

Find all vectors $\vec{v}_4$ in $R^4$ so that $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ form an orthonormal set.

Although you don't need it, the formula for the Gram-Schmidt process is

$$\vec{u}_i = \vec{v}_i - \left(\frac{< \vec{v}_i, \vec{u}_1 >}{< \vec{u}_1, \vec{u}_1 >}\right)\vec{u}_1 - \cdots - \left(\frac{< \vec{v}_i, \vec{u}_{i-1} >}{< \vec{u}_{i-1}, \vec{u}_{i-1} >}\right)\vec{u}_{i-1}$$

2. Since the given vectors are already orthonormal we look for $\vec{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ whose inner product with each of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is zero. This gives us the system of equtions

$$\begin{aligned} \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c + \frac{1}{2}d &= 0 \\ \frac{1}{2}a + \frac{1}{2}b - \frac{1}{2}c - \frac{1}{2}d &= 0 \\ \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}d &= 0 \end{aligned}$$

which has solution set $S = \left\{ d \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} : d \in \mathbf{C} \right\}$. The only vectors in this set that have norm equal to 1 are $\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$.

C.4. The matrix $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$ has the property that there is at least one vector $\overrightarrow{x}$ for which $A\overrightarrow{x} = 5\overrightarrow{x}$. Find all such vectors.

3. $A\vec{x} = 5\vec{x}$ can be rewritten as the system of equations

$$\begin{aligned} 4x - 2y &= 5x \\ -x + 3y &= 5y \end{aligned}$$

and this system can be rewritten as the homogenous system

$$\begin{aligned} -x - 2y &= 0 \\ -x - 2y &= 0 \end{aligned}$$

The solution set is: $S = \left\{ x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} : x_2 \in \mathbf{C} \right\}$.

**Do any two (2) of these problems from the text, homework, or class.**

**You may NOT just cite a theorem or result in the text. You must prove these results.**

M.1. Prove that if the matrix $A$ is nonsingular and $B$ is any appropriately sized matrix, then $N(AB) \subseteq N(B)$.

1. Let $\vec{x}$ be a vector in $N(AB)$ so that $AB(\vec{x}) = \vec{0}$. This means $A(B\vec{x}) = \vec{0}$ and since $A$ is nonsingular there is only the trivial solution to this matrix equation, namely, $B\vec{x} = \vec{0}$ which shows $\vec{x} \in N(B)$.

M.2. Prove DMAM (Distributivity across Matrix Addition): If $\alpha \in \mathbf{C}$, and $A, B \in M_{mn}$, then $\alpha(A + B) = \alpha A + \alpha B$.

2. Let $i$, $j$ be any indices satisfying $1 \leq i \leq m$, $1 \leq j \leq n$ then

$$\begin{aligned}
[\alpha (A+B)]_{ij} &= \alpha [A+B]_{ij} \\
&= \alpha \left( [A]_{ij} + [B]_{ij} \right) \\
&= \alpha [A]_{ij} + \alpha [B]_{ij} \\
&= [\alpha A]_{ij} + \alpha [B]_{ij} \\
&= [\alpha A + \alpha B]_{ij}
\end{aligned}$$

Since this equality holds for every entry entry of the two matrices, we have $\alpha (A+B) = \alpha A + \alpha B$.

M.3. Prove if $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is a linearly dependent set in $\mathbf{C}^{23}$, then the set

$$\{2\mathbf{w}_1 + \mathbf{w}_2 + 3\mathbf{w}_3, -3\mathbf{w}_1 + 2\mathbf{w}_2 + 4\mathbf{w}_3, \mathbf{w}_1 + 2\mathbf{w}_2 + 3\mathbf{w}_3\}$$

is linearly dependent.

3. Consider the relation of linear dependence

$$\beta_1 (2w_1 + w_2 + 3w_3) + \beta_2 (-3w_1 + 2w_2 + 4w_3) + \beta_3 (w_1 + 2w_2 + 3w_3) = \vec{0}$$

which, using distributibity, commutativity and associativity can be rewritten as

$$(2\beta_1 - 3\beta_2 + \beta_3) \mathbf{w}_1 + (\beta_1 + 2\beta_2 + 2\beta_3) \mathbf{w}_2 + (3\beta_1 + 4\beta_2 + 3\beta_3) \mathbf{w}_3 = \vec{0}$$

Since the set $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly dependent there is a nontrivial solution to this system of equations. That is, there are scalars $\alpha_1$, $\alpha_2$, $\alpha_3$, not all zero, that satisfy $\alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2 + \alpha_3 \mathbf{w}_3 = \mathbf{0}$. Since the system of equations

$$\begin{aligned}
2\beta_1 - 3\beta_2 + \beta_3 &= \alpha_1 \\
\beta_1 + 2\beta_2 + 2\beta_3 &= \alpha_2 \\
3\beta_1 + 4\beta_2 + 3\beta_3 &= \alpha_3
\end{aligned}$$

has a nonsingular coefficient matrix there is a unique solution. Since at least one of the $\alpha_i$ is not zero, the $i$th equation of the system shows that at least one of the $\beta_i$'s must not be zero. Hence there is a nontrivial relation of dependence

$$\beta_1 (2w_1 + w_2 + 3w_3) + \beta_2 (-3w_1 + 2w_2 + 4w_3) + \beta_3 (w_1 + 2w_2 + 3w_3) = \vec{0}$$

and the given set $\{2\mathbf{w}_1 + \mathbf{w}_2 + 3\mathbf{w}_3, -3\mathbf{w}_1 + 2\mathbf{w}_2 + 4\mathbf{w}_3, \mathbf{w}_1 + 2\mathbf{w}_2 + 3\mathbf{w}_3\}$ is linearly dependent.

## Do one (1) of these problems you've not seen before.

T.1. Suppose $A_{n\times m}$ and $B_{m\times n}$ are matrices such that $AB = I_n$. Let $\overrightarrow{b}$ be a particular vector in $R^n$. Show that the system of equations $A\overrightarrow{x} = \overrightarrow{b}$ must be consistent.

1. Since $(AB)\vec{b} = I_n\vec{b} = \vec{b}$ then by associativity $A\left(B\vec{b}\right) = \vec{b}$ and the vector $\vec{x} = B\vec{b}$ is a solution of the matrix equation $A\vec{x} = \vec{b}$. SOo the corresponding system of equations must be consistent.

T.2. Use the Principle of Mathematical Induction to prove that the statement $P(n)$ given by $\sum_{k=1}^{n} (2k-1) = n^2$ holds for all positive integers.

2. $P(1)$ is true since $\sum_{k=1}^{1} (2k-1) = (2-1) = 1 = 1^2$.
Suppose $P(n)$ is true. That is, $\sum_{k=1}^{n} (2k-1) = n^2$.
Then, $\sum_{k=1}^{n+1} (2k-1) = \left[\sum_{k=1}^{n} (2k-1)\right] + (2(n+1)-1) = n^2 + 2n + 1 = (n+1)^2$ and the truth of $P(n+1)$ follows from the truth of $P(n)$. Hence by the principle of mathematical induction, $\sum_{k=1}^{n} (2k-1) = n^2$ for every positive integer $n$.