Math 232   Spring 2006   Project 05

Due February 24

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Name

Be sure to re-read the WRITING GUIDELINES rubric, since it defines how your project will be graded. In particular, you may discuss this project with others but **you may not collaborate on the written exposition of the solution.**

"Never express yourself more clearly than you are able to think." — Niels Bohr

Geometric Model for Vectors: Rotations

One of the reasons we are studying abstract vectors is because they can be used to mathematically model a large number of important applications. The most common model is to interpret vectors in $\mathbf{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x, y \in \mathbf{R} \right\}$ as arrows in the Euclidean plane. Specifically, we interpret $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ as the set of arrows in the plane that have the same length and direction as the arrow that has it's base at the origin and its tip at the point (x, y). With this interpretation, the arrows represented by the sum of two vectors $\vec{u} + \vec{v}$ are obtained by putting the base of an arrow representing $\vec{v}$ at the tip of an arrow representing $\vec{u}$ and assigning the arrow with base the base of $\vec{u}$ and tip the tip of $\vec{v}$ to the sum $\vec{u} + \vec{v}$. The arrows representing $\alpha \vec{u}$ are obtained by stretching (or compressing) the length of $\vec{u}$ by a factor of α and keeping the same direction as $\vec{u}$ if α is positive and reversing the direction if α is negative.

The purpose of this project is to explore the linear algebra behind the operation of rotating, by an angle of θ, vectors in $\mathbf{R}^2$ about the origin. Specifically, let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a function that takes a vector $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ (an arrow with base at the origin and tip at the point (x, y)) as an input and outputs the vector $T(\vec{u})$ that is the arrow based at the origin and with tip the point resulting from rotating the point (x, y) **counterclockwise** around the origin by the angle θ.

If $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$, compute $T(\vec{u})$ by completing the following

1. Show the set of standard basis vectors $S = \{\vec{e}_1, \vec{e}_2\}$ satisfies $< S > = \mathbf{R}^2$.

2. Use a carefully drawn picture and the geometry of rotations to explain why $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$. [Do not compute the coordinates of the rotated vectors. Appeal to the geometry in your figure for justifications.]

3. Find $T(\vec{e}_1)$ and $T(\vec{e}_2)$.

4. Us all three of the above to compute $T(\vec{u})$ where $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Please note: this is probably the easiest way to compute the formula for rotating a vector around the origin when using orthogonal coordinates (we will have some slightly better notation later but the method would not change). On the other hand, if we think of these arrows as being in the **complex plane** so that we interpret the vector $\vec{u}$ as the complex number $x + iy = r(\cos(\phi) + i\sin(\phi)) = re^{i\phi}$ where r is the modulus of $\vec{u}$ and ϕ is the argument. Then using the polar coordinate system in the complex plane, the rotation of $\vec{u}$ by angle θ around the origin is given by a much simpler formula

$$T(\vec{u}) = T\left(re^{i\theta}\right) = re^{i(\phi+\theta)}.$$