Definitions, Axioms, Postulates, Propositions, and Theorems from *Euclidean and Non-Euclidean Geometries* by Marvin Jay Greenberg

Undefined Terms: Point, Line, Incident, Between, Congruent.

Incidence Axioms:

IA1: For every two distinct points there exists a unique line incident on them.

IA2: For every line there exist at least two points incident on it.

IA3: There exist three distinct points such that no line is incident on all three.

Incidence Propositions:

P2.1: If l and m are distinct lines that are non-parallel, then l and m have a unique point in common.

P2.2: There exist three distinct lines such that no point lies on all three.

P2.3: For every line there is at least one point not lying on it.

P2.4: For every point there is at least one line not passing through it.

P2.5: For every point there exist at least two distinct lines that pass through it.

Betweenness Axioms:

B1: If $A * B * C$, then A, B, and C are three distinct points all lying on the same line, and $C * B * A$.

B2: Given any two distinct points B and D, there exist points A, C, and E lying on $\overleftrightarrow{BD}$ such that $A{*}B{*}D$, $B{*}C{*}D$, and $B{*}D{*}E$.

B3: If A, B, and C are three distinct points lying on the same line, then one and only one of them is between the other two.

B4: For every line l and for any three points A, B, and C not lying on l:

1. If A and B are on the same side of l, and B and C are on the same side of l, then A and C are on the same side of l.
2. If A and B are on opposite sides of l, and B and C are on opposite sides of l, then A and C are on the same side of l.

Corollary If A and B are on opposite sides of l, and B and C are on the same side of l, then A and C are on opposite sides of l.

Betweenness Definitions:

Segment AB**:** Point A, point B, and all points P such that $A{*}P{*}B$.

Ray $\overrightarrow{AB}$**:** Segment AB and all points C such that $A{*}B{*}C$.

Line $\overleftrightarrow{AB}$**:** Ray $\overrightarrow{AB}$ and all points D such that $D{*}A{*}B$.

Same/Opposite Side: Let l be any line, A and B any points that do not lie on l. If $A = B$ or if segment AB contains no point lying on l, we say A and B are *on the same side of* l, whereas if $A \neq B$ and segment AB does intersect l, we say that A and B are *on opposite sides of* l. The law of excluded middle tells us that A and B are either on the same side or on opposite sides of l.

Betweenness Propositions:

P3.1: For any two points A and B:

1. $\overrightarrow{AB} \cap \overrightarrow{BA} = AB$, and
2. $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$.

P3.2: Every line bounds exactly two half-planes and these half-planes have no point in common.

Same Side Lemma: Given $A{*}B{*}C$ and l any line other than line $\overleftrightarrow{AB}$ meeting line $\overleftrightarrow{AB}$ at point A, then B and C are on the same side of line l.

Opposite Side Lemma: Given $A{*}B{*}C$ and l any line other than line $\overleftrightarrow{AB}$ meeting line $\overleftrightarrow{AB}$ at point B, then A and C are on opposite sides of line l.

P3.3: Given $A{*}B{*}C$ and $A{*}C{*}D$. Then $B{*}C{*}D$ and $A{*}B{*}D$.

P3.4: If $C{*}A{*}B$ and l is the line through A, B, and C, then for every point P lying on l, P either lies on ray $\overrightarrow{AB}$ or on the opposite ray $\overrightarrow{AC}$.

P3.5: Given $A{*}B{*}C$. Then $AC = AB \cup BC$ and B is the only point common to segments AB and BC.

P3.6: Given $A{*}B{*}C$. Then B is the only point common to rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$, and $\overrightarrow{AB} = \overrightarrow{AC}$.

Pasch's Theorem: If A, B, and C are distinct points and l is any line intersecting AB in a point between A and B, then l also intersects either AC, or BC. If C does not lie on l, then l does not intersect both AC and BC.

Angle Definitions:

Interior: Given an angle $\sphericalangle CAB$, define a point D to be in the *interior* of $\sphericalangle CAB$ if D is on the same side of $\overleftrightarrow{AC}$ as B and if D is also on the same side of $\overleftrightarrow{AB}$ as C. Thus, the interior of an angle is the intersection of two half-planes. (Note: the interior does not include the angle itself, and points not on the angle and not in the interior are on the exterior).

Ray Betweenness: Ray $\overrightarrow{AD}$ is *between* rays $\overrightarrow{AC}$ and $\overrightarrow{AB}$ provided $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not opposite rays and D is interior to $\sphericalangle CAB$.

Interior of a Triangle: The interior of a triangle is the intersection of the interiors of its thee angles. Define a point to be *exterior* to the triangle if it in not in the interior and does not lie on any side of the triangle.

Triangle: The union of the three segments formed by three non-collinear points.

Angle Propositions:

P3.7: Given an angle $\sphericalangle CAB$ and point D lying on line $\overleftrightarrow{BC}$. Then D is in the interior of $\sphericalangle CAB$ iff $B{*}D{*}C$.

"Problem 9": Given a line l, a point A on l and a point B not on l. Then every point of the ray $\overrightarrow{AB}$ (except A) is on the same side of l as B.

P3.8: If D is in the interior of $\sphericalangle CAB$, then:

1. so is every other point on ray $\overrightarrow{AD}$ except A,
2. no point on the opposite ray to $\overrightarrow{AD}$ is in the interior of $\sphericalangle CAB$, and
3. if $C{*}A{*}E$, then B is in the interior of $\sphericalangle DAE$.

P3.9:

1. If a ray r emanating from an exterior point of $\triangle ABC$ intersects side AB in a point between A and B, then r also intersects side AC or BC.
2. If a ray emanates from an interior point of $\triangle ABC$, then it intersects one of the sides, and if it does not pass through a vertex, then it intersects only one side.

Crossbar Theorem: If $\overrightarrow{AD}$ is between $\overrightarrow{AC}$ and $\overrightarrow{AB}$, then $\overrightarrow{AD}$ intersects segment BC.

Congruence Axioms:

C1: If A and B are distinct points and if A' is any point, then for each ray r emanating from A' there is a *unique* point B' on r such that $B' \neq A'$ and $AB \cong A'B'$.

C2: If $AB \cong CD$ and $AB \cong EF$, then $CD \cong EF$. Moreover, every segment is congruent to itself.

C3: If $A{*}B{*}C$, and $A'{*}B'{*}C'$, $AB \cong A'B'$, and $BC \cong B'C'$, then $AC \cong A'C'$.

C4: Given any $\sphericalangle BAC$ (where by definition of angle, $\overrightarrow{AB}$ is not opposite to $\overrightarrow{AC}$ and is distinct from $\overrightarrow{AC}$), and given any ray $\overrightarrow{A'B'}$ emanating from a point A', then there is a *unique* ray $\overrightarrow{A'C'}$ on a given side of line $\overleftrightarrow{A'B'}$ such that $\sphericalangle B'A'C' \cong \sphericalangle BAC$.

C5: If $\sphericalangle A \cong \sphericalangle B$ and $\sphericalangle A \cong \sphericalangle C$, then $\sphericalangle B \cong \sphericalangle C$. Moreover, every angle is congruent to itself.

C6 (SAS): If two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of another triangle, then the two triangles are congruent.

Congruence Propositions:

P3.10: If in $\triangle ABC$ we have $AB \cong AC$, then $\sphericalangle B \cong \sphericalangle C$.

P3.11: If $A*B*C$, $D*E*F$, $AB \cong DE$, and $AC \cong DF$, then $BC \cong EF$.

P3.12: Given $AC \cong DF$, then for any point B between A and C, there is a unique point E between D and F such that $AB \cong DE$.

P3.13:
1. Exactly one of the following holds: $AB < CD$, $AB \cong CD$, or $AB > CD$.
2. If $AB < CD$ and $CD \cong EF$, then $AB < EF$.
3. If $AB > CD$ and $CD \cong EF$, then $AB > EF$.
4. If $AB < CD$ and $CD < EF$, then $AB < EF$.

P3.14: Supplements of Congruent angles are congruent.

P3.15:
1. Vertical angles are congruent to each other.
2. An angle congruent to a right angle is a right angle.

P3.16: For every line l and every point P there exists a line through P perpendicular to l.

P3.17 (ASA): Given $\triangle ABC$ and $\triangle DEF$ with $\sphericalangle A \cong \sphericalangle D$, $\sphericalangle C \cong \sphericalangle F$, and $AC \cong DF$, then $\triangle ABC \cong \triangle DEF$.

P3.18: In in $\triangle ABC$ we have $\sphericalangle B \cong \sphericalangle C$, then $AB \cong AC$ and $\triangle ABC$ is isosceles.

P3.19: Given $\overrightarrow{BG}$ between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, $\overrightarrow{EH}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$, $\sphericalangle CBG \cong \sphericalangle FEH$ and $\sphericalangle GBA \cong \sphericalangle HED$. Then $\sphericalangle ABC \cong \sphericalangle DEF$.

P3.20: Given $\overrightarrow{BG}$ between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, $\overrightarrow{EH}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$, $\sphericalangle CBG \cong \sphericalangle FEH$ and $\sphericalangle ABC \cong \sphericalangle DEF$. Then $\sphericalangle GBA \cong \sphericalangle HED$.

P3.21:
1. Exactly one of the following holds: $\sphericalangle P < \sphericalangle Q$, $\sphericalangle P \cong \sphericalangle Q$, or $\sphericalangle P > \sphericalangle Q$.
2. If $\sphericalangle P < \sphericalangle Q$ and $\sphericalangle Q \cong \sphericalangle R$, then $\sphericalangle P < \sphericalangle R$.
3. If $\sphericalangle P > \sphericalangle Q$ and $\sphericalangle Q \cong \sphericalangle R$, then $\sphericalangle P > \sphericalangle R$.
4. If $\sphericalangle P < \sphericalangle Q$ and $\sphericalangle Q < \sphericalangle R$, then $\sphericalangle P < \sphericalangle R$.

P3.22 (SSS): Given $\triangle ABC$ and $\triangle DEF$. If $AB \cong DE$, $BC \cong EF$, and $AC \cong DF$, then $\triangle ABC \cong \triangle DEF$.

P3.23: All right angles are congruent to each other.

Corollary (not numbered in text) If P lies on l then the perpendicular to l through P is unique.

Definitions:

Segment Inequality: $AB < CD$ (or $CD > AB$) means that there exists a point E between C and D such that $AB \cong CE$.

Angle Inequality: $\sphericalangle ABC < \sphericalangle DEF$ means there is a ray $\overrightarrow{EG}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$ such that $\sphericalangle ABC \cong \sphericalangle GEF$.

Right Angle: An angle $\sphericalangle ABC$ is a right angle if has a supplementary angle to which it is congruent.

Parallel: Two lines l and m are parallel if they do not intersect, i.e., if no point lies on both of them.

Perpendicular: Two lines l and m are perpendicular if they intersect at a point A and if there is a ray $\overrightarrow{AB}$ that is a part of l and a ray $\overrightarrow{AC}$ that is a part of m such that $\sphericalangle BAC$ is a right angle.

Triangle Congruence and Similarity: Two triangles are congruent if a one-to-one correspondence can be set up between their vertices so that corresponding sides are congruent and corresponding angles are congruent. Similar triangles have this one-to-one correspondence only with their angles.

Circle (with center O and radius OA): The set of all points P such that OP is congruent to OA.

Triangle: The set of three distinct segments defined by three non-collinear points.

Continuity Axioms:

Archimedes' Axiom: If AB and CD are any segments, then there is a number n such that if segment CD is laid off n times on the ray $\overrightarrow{AB}$ emanating from A, then a point E is reached where $n \cdot CD \cong AE$ and B is between A and E.

Dedekind's Axiom: Suppose that the set of all points on a line l is the union $\Sigma_1 \cup \Sigma_2$ of two nonempty subsets such that no point of Σ_1 is between two points of Σ_2 and visa versa. Then there is a unique point O lying on l such that $P_1{*}O{*}P_2$ if and only if one of P_1, P_2 is in Σ_1, the other in Σ_2 and $O \neq P_1, P_2$. A pair of subsets Σ_1 and Σ_2 with the properties in this axiom is called a Dedekind cut of the line l.

Continuity Principles: Circular Continuity Principle: If a circle γ has one point inside and one point outside another circle γ', then the two circles intersect in two points.

Elementary Continuity Principle: In one endpoint of a segment is inside a circle and the other outside, then the segment intersects the circle.

Other Theorems, Propositions, and Corollaries in Neutral Geometry:

T4.1: If two lines cut by a transversal have a pair of congruent alternate interior angles, then the two lines are parallel.

Corollary 1: Two lines perpendicular to the same line are parallel. Hence the perpendicular dropped from a point P not on line l to l is unique.

Corollary 2: If l is any line and P is any point not on l, there exists at least one line m through P parallel to l.

T4.2 (Exterior Angle Theorem): An exterior angle of a triangle is greater than either remote interior angle.

T4.3 (see text for details): There is a unique way of assigning a degree measure to each angle, and, given a segment OI, called a unit segment, there is a unique way of assigning a length to each segment AB that satisfy our standard uses of angle and length.

Corollary 1: The sum of the degree measures of any two angles of a triangle is less than $180°$.

Corollary 2: If A, B, and C are three noncollinear points, then $\overline{AC} < \overline{AB} + \overline{BC}$.

T4.4 (Saccheri-Legendre): The sum of the degree measures of the three angles in any triangle is less than or equal to $180°$.

Corollary 1: The sum of the degree measures of two angles in a triangle is less than or equal to the degree measure of their remote exterior angle.

Corollary 2: The sum of the degree measures of the angles in any convex quadrilateral is at most $360°$ (**note: quadrilateral $\square ABCD$ is convex if it has a pair of opposite sides such that each is contained in a half-plane bounded by the other.**)

P4.1 (SAA): Given $AC \cong DF$, $\measuredangle A \cong \measuredangle D$, and $\measuredangle B \cong \measuredangle E$. Then $\triangle ABC \cong \triangle DEF$.

P4.2: Two right triangles are congruent if the hypotenuse and leg of one are congruent respectively to the hypotenuse and a leg of the other.

P4.3: Every segment has a unique midpoint.

P4.4:

1. Every angle has a unique bisector.
2. Every segment has a unique perpendicular bisector.

P4.5: In a triangle $\triangle ABC$, the greater angle lies opposite the greater side and the greater side lies opposite the greater angle, i.e., $AB > BC$ if and only if $\measuredangle C > \measuredangle A$.

P4.6: Given $\triangle ABC$ and $\triangle A'B'C'$, if $AB \cong A'B'$ and $BC \cong B'C'$, then $\measuredangle B < \measuredangle B'$ if and only if $AC < A'C'$.

Note: Statements up to this point are from or form neutral geometry. Choosing Hilbert's/Euclid's Axiom (the two are logically equivalent) or the Hyperbolic Axiom will make the geometry Euclidean or Hyperbolic, respectively.

Parallelism Axioms:

Hilbert's Parallelism Axiom for Euclidean Geometry: For every line l and every point P not lying on l there is at most one line m through P such that m is parallel to l. (Note: it can be proved from the previous axioms that, assuming this axiom, there is **EXACTLY** one line m parallel to l [see T4.1 Corollary 2]).

Euclid's Fifth Postulate: If two lines are intersected by a transversal in such a way that the sum of the degree measures of the two interior angles on one side of the transversal is less than 180°, then the two lines meet on that side of the transversal.

Hyperbolic Parallel Axiom: There exist a line l and a point P not on l such that at least two distinct lines parallel to l pass through P.

Hilbert's Parallel Postulate is logically equivalent to the following:

T4.5: Euclid's Fifth Postulate.

P4.7: If a line intersects one of two parallel lines, then it also intersects the other.

P4.8: Converse to Theorem 4.1.

P4.9: If t is transversal to l and m, $l \| m$, and $t \perp l$, then $t \perp m$.

P4.10: If $k \| l$, $m \perp k$, and $n \perp l$, then either $m = n$ or $m \| n$.

P4.11: The angle sum of every triangle is 180°.

Wallis: Given any triangle $\triangle ABC$ and given any segment DE. There exists a triangle $\triangle DEF$ (having DE as one of its sides) that is similar to $\triangle ABC$ (denoted $\triangle DEF \sim \triangle ABC$).

Theorems 4.6 and 4.7 (see text) are used to prove P4.11. They define the *defect* of a triangle to be the 180° minus the angle sum, then show that if one defective triangle exists, then all triangles are defective. Or, in contrapositive form, if one triangle has angle sum 180°, then so do all others. They do not assume a parallel postulate.

Theorems Using the Parallel Axiom

Parallel Projection Theorem: Given three parallel lines l, m, and n. Let t and t' be transversals to these parallels, cutting them in points A, B, and C and in points A', B', and C', respectively. Then $\overline{AB}/\overline{BC} = \overline{A'B'}/\overline{B'C'}$.

Fundamental Theorem on Similar Triangles: Given $\triangle ABC \sim \triangle A'B'C'$. Then the corresponding sides are proportional.

HYPERBOLIC GEOMETRY

L6.1: There exists a triangle whose angle sum is less than 180°.

Universal Hyperbolic Theorem: In hyperbolic geometry, from every line l and every point P not on l there pass through P at least two distinct parallels to l.

T6.1: Rectangles do not exist and all triangles have angle sum less than 180°.

Corollary: In hyperbolic geometry, all convex quadrilaterals have angle sum less than 360°.

T6.2: If two triangles are similar, they are congruent.

T6.3: If l and l' are any distinct parallel lines, then any set of points on l equidistant from l' has at most two points in it.

T6.4: If l and l' are parallel lines for which there exists a pair of points A and B on l equidistant from l', then l and l' have a common perpendicular segment that is also the shortest segment between l and l'.

L6.2: The segment joining the midpoints of the base and summit of a Saccheri quadrilateral is perpendicular to both the base and the summit, and this segment is shorter than the sides.

T6.5: If lines l and l' have a common perpendicular MM', then they are parallel and MM' is unique. Moreover, if A and B are points on l such that M is the midpoint of segment AB, then A and B are equidistant from l'.

T6.6: For every line l and every point P not on l, let Q be the foot of the perpendicular from P to l. Then there are two unique rays $\overrightarrow{PX}$ and $\overrightarrow{PX'}$ on opposite sides of $\overleftrightarrow{PQ}$ that do not meet l and have the property that a ray emanating from P meets l if and only if it is between $\overrightarrow{PX}$ and $\overrightarrow{PX'}$. Moreover, these limiting rays are situated symmetrically about $\overleftrightarrow{PQ}$ in the sense that $\measuredangle XPQ \cong \measuredangle X'PQ$.

T6.7: Given m parallel to l such that m does not contain a limiting parallel ray to l in either direction. Then there exists a common perpendicular to m and l, which is unique.

Results from chapter 7 (Contextual definitions not included):

P7.1
1. $P = P'$ if and only if P lies on the circle of inversion γ.
2. If P is inside γ then P' is outside γ, and if P is outside γ, then P' is inside γ.
3. $(P')' = P$.

P7.2 Suppose P is inside γ. Let TU be the chord of γ which is perpendicular to $\overleftrightarrow{OP}$. Then the inverse P' of P is the pole of chord TU, i.e., the point of intersection of the tangents to γ at T and U.

P7.3 If P is outside γ, let Q be the midpoint of segment OP. Let σ be the circle with center Q and radius $\overline{OQ} = \overline{QP}$. Then σ cuts γ in two points T and U, $\overleftrightarrow{PT}$ and $\overleftrightarrow{PU}$ are tangent to γ, and the inverse P' of P is the intersection of TU and OP.

P7.4 Let T and U be points on γ that are not diametrically opposite and let P be the pole of TU. Then $PT \cong PU$, $\measuredangle PTU \cong \measuredangle PUT$, $\overleftrightarrow{OP} \perp \overleftrightarrow{TU}$, and the circle δ with center P and radius $\overline{PT} = \overline{PU}$ cuts γ orthogonally at T and U.

L7.1 Given that point O does not lie on circle δ.

1. If two lines through O intersect δ in pairs of points (P_1, P_2) and (Q_1, Q_2), respectively, then we have $(\overline{OP_1})(\overline{OP_2}) = (\overline{OQ_1})(\overline{OQ_2})$. This common product is called the *power* of O with respect to δ when O is outside of δ, and minus this number is called the power of O when O is inside δ.
2. If O is outside δ and a tangent to δ from O touches δ at point T, then $(\overline{OT})^2$ equals the power of O with respect to δ.

P7.5 Let P be any point which does not lie on circle γ and which does not coincide with the center O of γ, and let δ be a circle through P. Then δ cuts γ orthogonally if and only if δ passes through the inverse point P' of P with respect to γ.