Definitions, Axioms, Postulates, Propositions, and Theorems from *Euclidean and Non-Euclidean Geometries, 4th Ed* by Marvin Jay Greenberg (Revised: 18 Feb 2011)

Logic Rule 0 No unstated assumptions may be used in a proof.

Logic Rule 1 Allowable justifications.

1. "By hypothesis ...".
2. "By axiom ...".
3. "By theorem ..." (previously proved).
4. "By definition ...".
5. "By step ..." (a previous step in the argument).
6. "By rule ..." of logic.

Logic Rule 2 Proof by contradiction (RAA argument).

Logic Rule 3 The tautology $\sim(\sim S) \iff S$

Logic Rule 4 The tautology $\sim(H \implies C) \iff H \wedge (\sim C)$.

Logic Rule 5 The tautology $\sim(S_1 \wedge S_2) \iff (\sim S_1 \vee \sim S_2)$.

Logic Rule 6 The statement $\sim(\forall x S(x))$ means the same as $\exists x(\sim S(x))$.

Logic Rule 7 The statement $\sim(\exists x S(x))$ means the same as $\forall x(\sim S(x))$.

Logic Rule 8 The tautology $((P \implies Q) \wedge P) \implies Q$.

Logic Rule 9 The tautologies

1. $((P \implies Q) \wedge (Q \implies R) \implies (P \implies R)$.
2. $(P \wedge Q) \implies P$ and $(P \wedge Q) \implies Q$.
3. $(\sim Q \implies \sim P) \implies ((P \implies Q)$.

Logic Rule 10 The tautology $P \implies (P \vee \sim P)$.

Logic Rule 11 (Proof by Cases) If C can be deduced from each of $S_1, S_2, \cdots, S_n$ **individually**, then $(S_1 \vee S_2 \vee \cdots S_n) \implies C$ is a tautology.

Logic Rule 12 Euclid's "Common Notions"

1. $\forall X\ (X = X)$
2. $\forall X\ \forall Y\ (X = Y \iff Y = X)$
3. $\forall X\ \forall Y\ \forall Z\ ((X = Y \wedge Y = Z) \implies X = Z)$
4. If $X = Y$ and $S(X)$ is a statement about X, then $S(X) \iff S(Y)$

Undefined Terms: Point, Line, Incident, Between, Congruent.

Basic Definitions

1. Three or more points are **collinear** if there exists a line incident with all of them.
2. Three or more lines are **concurrent** if there is a point incident with all of the them.
3. Two lines are **parallel** if they are distinct and no point is incident with both of them.
4. $\{\overleftrightarrow{AB}\}$ is the set of points incident with $\overleftrightarrow{AB}$.

Incidence Axioms:

IA1: For every two distinct points there exists a unique line incident on them.

IA2: For every line there exist at least two points incident on it.

IA3: There exist three distinct points such that no line is incident on all three.

Incidence Propositions:

P2.1: If l and m are distinct lines that are not parallel, then l and m have a unique point in common.

P2.2: There exist three distinct lines that are not concurrent.

P2.3: For every line there is at least one point not lying on it.

P2.4: For every point there is at least one line not passing through it.

P2.5: For every point there exist at least two distinct lines that pass through it.

P2.6: For every point P there are at least two distinct points neither of which is P.

P2.7: For every line l there are at least two distinct lines neither of which is l.

P2.8: If l is a line and P is a point not incident with l then there is a one-to-one correspondence between the set of points incident with l and the set of lines through P that meet l.

P2.9: Let P be a point. Denote the set of points $\{X : X \text{ is on a line passing through } P\}$ by S. Then every point is in S.

P2.10: Let l be a line. Denote the set of points $\{m : m \text{ is incident with a point that lies on } l \text{ or } m \text{ is parallel to } l\}$ by S. Then every point is in S.

Betweenness Axioms and Notation:

Notation: $A * B * C$ means "point B is between point A and point C."

B1: If $A * B * C$, then A, B, and C are three distinct points all lying on the same line, and $C * B * A$.

B2: Given any two distinct points B and D, there exist points A, C, and E lying on $\overleftrightarrow{BD}$ such that $A*B*D$, $B*C*D$, and $B * D * E$.

B3: If A, B, and C are three distinct points lying on the same line, then one and only one of them is between the other two.

Lemma LPD (Line-Point Decomposition) Let X be a point on line $\overleftrightarrow{AB}$. Then exactly one of the following holds: $X = A,\ X = B,\ X * A * B,\ A * X * B,\ A * B * X.$

B4: For every line l and for any three points A, B, and C not lying on l:

1. If A and B are on the same side of l, and B and C are on the same side of l, then A and C are on the same side of l.
2. If A and B are on opposite sides of l, and B and C are on opposite sides of l, then A and C are on the same side of l.

Corollary If A and B are on opposite sides of l, and B and C are on the same side of l, then A and C are on opposite sides of l.

Betweenness Definitions:

Segment AB: Point A, point B, and all points P such that $A * P * B$.

Ray $\overrightarrow{AB}$: Segment AB and all points C such that $A * B * C$.

Same/Opposite Side: Let l be any line, A and B any points that do not lie on l. If $A = B$ or if segment AB contains no point lying on l, we say A and B are *on the same side of* l, whereas if $A \neq B$ and segment AB does intersect l, we say that A and B are *on opposite sides of* l. The law of excluded middle tells us that A and B are either on the same side or on opposite sides of l.

Betweenness Propositions:

P3.1 (does not use BA-4): For any two points A and B:

1. $\overrightarrow{AB} \cap \overrightarrow{BA} = AB$ **Proof is in text**, and
2. $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$.

P3.2 Proof is in text: Every line bounds exactly two half-planes and these half-planes have no point in common.

Same Side Lemma: Given $A * B * C$ and l any line other than line $\overleftrightarrow{AB}$ meeting line $\overleftrightarrow{AB}$ at point A, then B and C are on the same side of line l.

Opposite Side Lemma: Given $A * B * C$ and l any line other than line $\overleftrightarrow{AB}$ meeting line $\overleftrightarrow{AB}$ at point B, then A and C are on opposite sides of line l.

P3.3 Proof is in text: Given $A * B * C$ and $A * C * D$. Then $B * C * D$ and $A * B * D$.

Corollary to P3.3: Given $A * B * C$ and $B * C * D$. Then $A * B * D$ and $A * C * D$.

P3.4 Proof is in text: If $C * A * B$ and l is the line through A, B, and C, then for every point P lying on l, P either lies on ray $\overrightarrow{AB}$ or on the opposite ray $\overrightarrow{AC}$.

Pasch's Theorem Proof is in text: If A, B, and C are distinct noncollinear points and l is any line intersecting AB in a point between A and B, then l also intersects either AC, or BC. If C does not lie on l, then l does not intersect both AC and BC.

P3.5: Given $A * B * C$. Then $AC = AB \cup BC$ and B is the only point common to segments AB and BC.

P3.6: Given $A * B * C$. Then B is the only point common to rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$, and $\overrightarrow{AB} = \overrightarrow{AC}$.

Angle Definitions:

Interior: (Occurs after P3.6) Given an angle $\sphericalangle CAB$, define a point D to be in the *interior* of $\sphericalangle CAB$ if D is on the same side of $\overleftrightarrow{AC}$ as B and if D is also on the same side of $\overleftrightarrow{AB}$ as C. Thus, the interior of an angle is the intersection of two half-planes. (Note: the interior does not include the angle itself, and points not on the angle and not in the interior are on the exterior).

Ray Betweenness: (Occurs after P3.8) Ray $\overrightarrow{AD}$ is *between* rays $\overrightarrow{AC}$ and $\overrightarrow{AB}$ provided $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not opposite rays and D is interior to $\sphericalangle CAB$.

Triangle: (Occurs after Ch2) The union of the three segments formed by three non-collinear points.

Interior of a Triangle: (Occurs after Crossbar Thm) The interior of a triangle is the intersection of the interiors of its thee angles. Define a point to be *exterior* to the triangle if it in not in the interior and does not lie on any side of the triangle.

Angle Propositions:

P3.7: Given an angle $\sphericalangle CAB$ and point D lying on line $\overleftrightarrow{BC}$. Then D is in the interior of $\sphericalangle CAB$ iff $B * D * C$.

"Problem 9": Given a line l, a point A on l and a point B not on l. Then every point of the ray $\overrightarrow{AB}$ (except A) is on the same side of l as B.

P3.8: If D is in the interior of $\sphericalangle CAB$, then:

1. so is every other point on ray $\overrightarrow{AD}$ except A,
2. no point on the opposite ray to $\overrightarrow{AD}$ is in the interior of $\sphericalangle CAB$, and
3. if $C * A * E$, then B is in the interior of $\sphericalangle DAE$.

Crossbar Theorem: If $\overrightarrow{AD}$ is between $\overrightarrow{AC}$ and $\overrightarrow{AB}$, then $\overrightarrow{AD}$ intersects segment BC.

P3.9:

1. If a ray r emanating from an exterior point of $\triangle ABC$ intersects side AB in a point between A and B, then r also intersects side AC or BC.
2. If a ray emanates from an interior point of $\triangle ABC$, then it intersects one of the sides, and if it does not pass through a vertex, then it intersects only one side.

Congruence Axioms:

C1: If A and B are distinct points and if A' is any point, then for each ray r emanating from A' there is a *unique* point B' on r such that $B' \neq A'$ and $AB \cong A'B'$.

C2: If $AB \cong CD$ and $AB \cong EF$, then $CD \cong EF$. Moreover, every segment is congruent to itself.

C3: If $A * B * C$, and $A' * B' * C'$, $AB \cong A'B'$, and $BC \cong B'C'$, then $AC \cong A'C'$.

C4: Given any $\sphericalangle BAC$ (where by definition of angle, $\overrightarrow{AB}$ is not opposite to $\overrightarrow{AC}$ and is distinct from $\overrightarrow{AC}$), and given any ray $\overrightarrow{A'B'}$ emanating from a point A', then there is a *unique* ray $\overrightarrow{A'C'}$ on a given side of line $\overleftrightarrow{A'B'}$ such that $\sphericalangle B'A'C' \cong \sphericalangle BAC$.

C5: If $\sphericalangle A \cong \sphericalangle B$ and $\sphericalangle A \cong \sphericalangle C$, then $\sphericalangle B \cong \sphericalangle C$. Moreover, every angle is congruent to itself.

C6 (SAS): If two sides and the included angle of one triangle are congruent, respectively, to two sides and the included angle of another triangle, then the two triangles are congruent.

Congruence Propositions:

Corollary to SAS: Proof is in text Given $\triangle ABC$ and segment $DE \cong AB$, there is a unique point F on a given side of line $\overleftrightarrow{DE}$ such that $\triangle ABC \cong \triangle DEF$.

P3.10 Proof is in text: If in $\triangle ABC$ we have $AB \cong AC$, then $\sphericalangle B \cong \sphericalangle C$.

P3.11: (Segment Subtraction) If $A * B * C$, $D * E * F$, $AB \cong DE$, and $AC \cong DF$, then $BC \cong EF$.

P3.12 Proof is in text: Given $AC \cong DF$, then for any point B between A and C, there is a unique point E between D and F such that $AB \cong DE$.

P3.13: (Segment Ordering)

1. Exactly one of the following holds: $AB < CD$, $AB \cong CD$, or $AB > CD$.
2. If $AB < CD$ and $CD \cong EF$, then $AB < EF$.
3. If $AB > CD$ and $CD \cong EF$, then $AB > EF$.
4. If $AB < CD$ and $CD < EF$, then $AB < EF$.

P3.14: Supplements of congruent angles are congruent.

P3.15:
1. Vertical angles are congruent to each other.
2. An angle congruent to a right angle is a right angle.

P3.16 Proof is in text: For every line l and every point P there exists a line through P perpendicular to l.

P3.17 (ASA): Given $\triangle ABC$ and $\triangle DEF$ with $\sphericalangle A \cong \sphericalangle D$, $\sphericalangle C \cong \sphericalangle F$, and $AC \cong DF$, then $\triangle ABC \cong \triangle DEF$.

P3.18: If in $\triangle ABC$ we have $\sphericalangle B \cong \sphericalangle C$, then $AB \cong AC$ and $\triangle ABC$ is isosceles.

P3.19 Proof is in text: (Angle Addition) Given $\overrightarrow{BG}$ between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, $\overrightarrow{EH}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$, $\sphericalangle CBG \cong \sphericalangle FEH$ and $\sphericalangle GBA \cong \sphericalangle HED$. Then $\sphericalangle ABC \cong \sphericalangle DEF$.

P3.20: (Angle Subtraction) Given $\overrightarrow{BG}$ between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, $\overrightarrow{EH}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$, $\sphericalangle CBG \cong \sphericalangle FEH$ and $\sphericalangle ABC \cong \sphericalangle DEF$. Then $\sphericalangle GBA \cong \sphericalangle HED$.

P3.21: (Ordering of Angles)

1. Exactly one of the following holds: $\sphericalangle P < \sphericalangle Q$, $\sphericalangle P \cong \sphericalangle Q$, or $\sphericalangle P > \sphericalangle Q$.
2. If $\sphericalangle P < \sphericalangle Q$ and $\sphericalangle Q \cong \sphericalangle R$, then $\sphericalangle P < \sphericalangle R$.
3. If $\sphericalangle P > \sphericalangle Q$ and $\sphericalangle Q \cong \sphericalangle R$, then $\sphericalangle P > \sphericalangle R$.
4. If $\sphericalangle P < \sphericalangle Q$ and $\sphericalangle Q < \sphericalangle R$, then $\sphericalangle P < \sphericalangle R$.

P3.22 (SSS): Given $\triangle ABC$ and $\triangle DEF$. If $AB \cong DE$, $BC \cong EF$, and $AC \cong DF$, then $\triangle ABC \cong \triangle DEF$.

P3.23 Proof is in text: (Euclid's Fourth Postulate) All right angles are congruent to each other.

Corollary (not numbered in text) If P lies on l then the perpendicular to l through P is unique.

Definitions:

Segment Inequality: $AB < CD$ (or $CD > AB$) means that there exists a point E between C and D such that $AB \cong CE$.

Angle Inequality: $\sphericalangle ABC < \sphericalangle DEF$ means there is a ray $\overrightarrow{EG}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$ such that $\sphericalangle ABC \cong \sphericalangle GEF$.

Right Angle: An angle $\sphericalangle ABC$ is a right angle if has a supplementary angle to which it is congruent.

Parallel: Two lines l and m are parallel if they do not intersect, i.e., if no point lies on both of them.

Perpendicular: Two lines l and m are perpendicular if they intersect at a point A and if there is a ray $\overrightarrow{AB}$ that is a part of l and a ray $\overrightarrow{AC}$ that is a part of m such that $\sphericalangle BAC$ is a right angle.

Triangle Congruence and Similarity: Two triangles are congruent if a one-to-one correspondence can be set up between their vertices so that corresponding sides are congruent and corresponding angles are congruent. Similar triangles have this one-to-one correspondence only with their angles.

Circle (with center O and radius OA): The set of all points P such that OP is congruent to OA.

Triangle: The set of three distinct segments defined by three non-collinear points.

Acute, Obtuse Angles An angle is *acute* if it is less than a right angle, *obtuse* if it is greater than a right angle.

Hilbert Plane A model of our incidence, betweenness, and congruence axioms is called a *Hilbert Plane*.

Continuity Axioms and Principles:

Circle-Circle Continuity Principle If a circle γ has one point inside and one point outside another circle γ', then the two circles intersect in two points.

Line-Circle Continuity Principle If a line passes through a point inside a circle, then the line intersects the circle in two points.

Segment-Circle Continuity Principle In one endpoint of a segment is inside a circle and the other outside, then the segment intersects the circle at a point in between.

Archimedes' Axiom: If CD is any segment, A and point, and r any ray with vertex A, then for every point $B \neq A$ on r there is a number n such that when CD is laid off nn times on r starting at A, a point E is reached such that $n \cdot CD \cong AE$ and either $B = E$ or B is between A and E.

Aristotle's Angle Unboundedness Axiom Given and side of an acute angle and any segment AB, there exists a point Y on the given side if the angle such that if X is the foot of the perpendicular from Y to the other side of the angle, $XY > AB$.

Important Corollary to Aristotle's Axiom Let $\overrightarrow{AB}$ be any ray, P any point not collinear with A and B, and $\sphericalangle\ XVY$ any acute angle. Then there exists a point R on ray $\overrightarrow{AB}$ such that $\sphericalangle\ PRA < \sphericalangle\ XVY$.

Dedekind's Axiom: Suppose that the set of all points on a line l is the union $\Sigma_1 \cup \Sigma_2$ of two nonempty subsets such that no point of either is between two points of the other. Then there is a unique point O lying on l such that one of the subsets is equal to a ray of l with vertex O and the other subset is equal to the complement.

Hilbert's Euclidean Axiom of Parallelism For every line l and every point P not lying on l there is a at most one line m through P such that m is parallel to l.

Definition of Euclidean Plane A *Euclidean Plane* is a Hilbert Plane in which Hilbert's Euclidean axiom of parallelism and the circle-circle continuity principle hold.

Theorems, Propositions, and Corollaries in Neutral Geometry:

T4.1 (AIA) Proof is in text: In any Hilbert plane, if two lines cut by a transversal have a pair of congruent alternate interior angles with respect to that transversal, then the two lines are parallel.

Corollary 1 Proof is in text: Two lines perpendicular to the same line are parallel. Hence the perpendicular dropped from a point P not on line l to l is unique.

Corollary 2 (Euclid I.31.) Proof is in text: If l is any line and P is any point not on l, there exists at least one line m through P parallel to l.

T4.2 (EA) Proof is in text: In any Hilbert plane, an exterior angle of a triangle is greater than either remote interior angle.

Corollary 1 to EA If a triangle has a right or obtuse angle, the other two angles are acute.

Corollary 2 to EA (requires Theorem 4.3) The sum of the degree measures of any two angles of a triangle is less than $180°$.

P4.1 (SAA): Given $AC \cong DF$, $\sphericalangle\ A \cong \sphericalangle\ D$, and $\sphericalangle\ B \cong \sphericalangle\ E$. Then $\triangle ABC \cong \triangle DEF$.

P4.2 (Hypotenuse-Leg): Two right triangles are congruent if the hypotenuse and leg of one are congruent respectively to the hypotenuse and a leg of the other.

P4.3 (Midpoints): Every segment has a unique midpoint.

P4.4 (Bisectors):

1. Every angle has a unique bisector.
2. Every segment has a unique perpendicular bisector.

P4.5: In a triangle $\triangle ABC$, the greater angle lies opposite the greater side and the greater side lies opposite the greater angle, i.e., $AB > BC$ if and only if $\sphericalangle\ C > \sphericalangle\ A$.

P4.6: Given $\triangle ABC$ and $\triangle A'B'C'$, if $AB \cong A'B'$ and $BC \cong B'C'$, then $\sphericalangle\ B < \sphericalangle\ B'$ if and only if $AC < A'C'$.

T4.3 Measurement Theorem (see text for details): There is a unique way of assigning a degree measure to each angle, and, given a segment OI, called a unit segment, there is a unique way of assigning a length to each segment AB that satisfy our standard uses of angle and length.

Corollary 2 to EA Theorem Proof is in text: The sum of the degree measures of any two angles of a triangle is less than 180°.

Triangle Inequality Proof is in text: If $\overline{AB}, \overline{BC}, \overline{AC}$ are the lengths of the sides of a triangle $\triangle ABC$, then $\overline{AC} < \overline{AB} + \overline{BC}$.

Corollary For any Hilbert plane, the converse to the triangle inequality is equivalent to the circle-circle continuity principle. Hence the converse to the triangle inequality holds in Euclidean planes.

Note: Statements up to this point are from neutral geometry. Choosing Hilbert's/Euclid's Axiom (the two are logically equivalent) or the Hyperbolic Axiom will make the geometry Euclidean or Hyperbolic, respectively.

Parallelism Axioms:

Hilbert's Parallelism Axiom for Euclidean Geometry: For every line l and every point P not lying on l there is at most one line m through P such that m is parallel to l. (Note: it can be proved from the previous axioms that, assuming this axiom, there is **EXACTLY** one line m parallel to l [see T4.1 Corollary 2]).

Euclid's Fifth Postulate: If two lines are intersected by a transversal in such a way that the sum of the degree measures of the two interior angles on one side of the transversal is less than 180°, then the two lines meet on that side of the transversal.

Hyperbolic Parallel Axiom: There exist a line l and a point P not on l such that at least two distinct lines parallel to l pass through P.

Equivalences to Hilbert's Parallel Postulate (HPP):

T4.4 Proof is in text: Euclid's Fifth Postulate $\iff$ Hilbert's Euclidean parallel postulate.

P4.7: If a line intersects one of two parallel lines, then it also intersects the other $\iff$ HPP.

P4.8: Converse to Alternate Interior Angle Theorem $\iff$ HPP.

P4.9: If t is transversal to l and m, $l \parallel m$, and $t \perp l$, then $t \perp m \iff$ HPP.

P4.10: If $k \parallel l$, $m \perp k$, and $n \perp l$, then either $m = n$ or $m \parallel n \iff$ HPP.

P4.11: In any Hilbert plane, Hilbert's Euclidean parallel postulate implies that for every triangle $\triangle ABC$ the angle sum 180°.

Corollary Hilbert's Euclidean parallel postulate implies that the degree of an exterior angle to a triangle is equal to the sum of the degrees of its remote interior angles.

Proposition 4.12 Proof is in text

1. (Saccheri I). The summit angles of a Saccheri quadrilateral are congruent to each other.
2. (Saccheri II). The line joining the midpoints of the summit and base is perpendicular to both the summit and base.

Definitions Bi-right quadrilateral A quadrilateral $\square ABDC$ in which the adjacent angles $\sphericalangle A$ and $\sphericalangle B$ are right angles.

Saccheri quadrilateral A bi-right quadrilateral $\square ABDC$ in which sides CA and DB are congruent.

Lambert quadrilateral A quadrilateral with at least three right angles.

semi-Euclidean Hilbert Plane A Hilbert plane is semi-Euclidean if all Lambert quadrilaterals and all Saccheri quadrilaterals are rectangles. In addition, if the fourth angle of every Lambert quadrilateral is acute (respectively, obtuse), we say the plane satisfies the acute (respectively, obtuse0 angle hypothesis.

Convex quadrilateral A quadrilateral $\square ABCD$ which has a pair of opposite sides, e.g., AB and CD, such that CD is contained in a half-plane bounded by $\overleftrightarrow{AB}$ and AB is contained in a half-plane bounded by $\overleftrightarrow{CD}$.

Proposition 4.13 Proof is in text: In any bi-right quadrilateral $\square ABDC$, $\sphericalangle C > \sphericalangle D \Leftrightarrow BD > AC$. "The greater side is opposite the greater angle."

Corollary 1 Proof is in text: Given any acute angle with vertex V. Let Y be any point on one side of the angle, let Y' be any point satisfying $V * Y * Y'$. Let X, X' be the feet of the perpendiculars from Y, Y', respectively, to the other side of the angle. Then $Y'X' > YX$. "Perpendicular segments from one side of an acute angle to the other increases s you move away from the vertex of the angle."

Corollary 2 Proof is in text: Euclid V implies Arisototle's Axiom.

Corollary 3 Proof is in text: A side adjacent to the fourth angle θ of a Lambert quadrilateral is, respectively, greater than, congruent to or less than its opposite side if and only if θ is acute, right, or obtuse, respectively.

Corollary 4 Proof is in text: The summit of a Saccheri quadrilateral is, respectively, greater than, congruent to, or less than the base if and only if its summit angle is acute, right, or obtuse, respectively.

Uniformity Theorem: For any Hilbert plane, if one Saccheri quadrilateral has acute (respectively, right, obtuse) summit angles, then so do all Saccheri quadrilaterals.

Corollary 1 For any Hilbert plane, if one Lambert quadrilateral has an acute (respectively, right, obtuse) fourth angle, then so do all Lambert quadrilaterals. Furthermore, the type of the fourth angle is the same as the type of the summit angles of Saccheri quadrilaterals.

Corollary 2 There exists a rectangle in a Hilbert plane iff the plane is semi-Euclidean. Opposite sides of a rectangle are congruent to each other.

Corollary 3 In a Hilbert plane satisfying the acute (respectively, obtuse) angle hypothesis, a side of a Lambert quadrilateral adjacent to the acute (respectively, obtuse) angle is greater than (respectively, less than) its opposite side.

Corollary 4 In a Hilbert plane satisfying the acute (respectively, obtuse) angle hypothesis, the summit of a Saccheri quadrilateral is greater than (respectively, less than) the base. The midline segment MN is the only common perpendicular segment between the summit and the base line. If P is any point other than M on the summit line and Q is the foot of the perpendicular to the base line, then $PQ > MN$ (respectively, $PQ < MN$. As P moves away from M along a ray of the summit line emanating from M, PQ increases (respectively, decreases).

Definitions: Angle sum of a triangle The angle sum of triangle $\triangle ABC$ is the sum of the degree measures of the three angles of the triangle.

Defect of a triangle The defect, $\delta(ABC)$, of triangle $\triangle ABC$ is $180°$ minus the angle sum.

Saccheri's Angle Theorem Proof is in text For any Hilbert Plane

1. If there exists a triangle whose angle sum is $< 180°$, then every triangle has an angle sum $< 180°$, and this is equivalent to the fourth angles of Lambert quadrilaterals and the summit angles of Saccheri quadrilaterals being acute.
2. If there exists a triangle with angle sum $= 180°$, then every triangle has angle sum $= 180°$, and this is equivalent to the plane being semi-Euclidean.
3. If there exists a triangle whose angle sum is $> 180°$, then every triangle has an angle sum $> 180°$, and this is equivalent to the fourth angles of Lambert quadrilaterals and the summit angles of Saccheri quadrilaterals being obtuse.

Lemma Proof is in text Let $\square ABDC$ be a Saccheri quadrilateral with summit angle class θ. Consider the alternate interior angles $\sphericalangle ACB$ and $\sphericalangle DBC$ with respect to diagonal CB.

1. $\sphericalangle ACB < \sphericalangle DBC$ iff θ is acute.
2. $\sphericalangle ACB \cong \sphericalangle DBC$ iff θ is right.
3. $\sphericalangle ACB > \sphericalangle DBC$ iff θ is obtuse.

Non-Obtuse-Angle Theorem Proof is in text A Hilbert plane satisfying Aristotle's axiom either is semi-Euclidean or satisfies the acute angle hypothesis (which implies the angle sum of every triangle is $< 180°$.

Corollary In a Hilbert plane satisfying Aristotle's axiom, an exterior angle of a triangle is greater than or congruent to the sum of the two remote interior angles.

Saccheri-Legendre Theorem In an Archimedean Hilbert plane, the angle sum of every triangle is $< 180°$.

Results from Chapter 5 Clavius' Axiom For any line l and any point P not on l, the equidistant locus to l through P is the set of all the points on a line through P (which is parallel to l).

Theorem (about Clavius' in neutral geometry) The following three statements are equivalent for a Hilbert plane:

1. The plane is semi-Euclidean.
2. For any line l and any point P not on l, the equidistant locus to l through P is the set of all the points on the parallel to l through P obtained by the standard construction.
3. Clavius' Axiom.

Wallis: Given any triangle $\triangle ABC$ and given any segment DE. There exists a triangle $\triangle DEF$ (having DE as one of its sides) that is similar to $\triangle ABC$ (denoted $\triangle DEF \sim \triangle ABC$). This statement is equivalent to Euclid V in neutral geometry.

Claraut's Axiom: Rectangles exist.

Proclus' Theorem the Euclidean parallel postulate holds in a Hilbert plane if and only if the plane is semi-Euclidean (i.e. the angle sum is 180°) and Aristotle's angle unboundedness axiom holds.

Legendre's Theorem (still assuming Archimedes Axiom Hypothesis: For any acute angle $\sphericalangle A$ and any point D in the interior of $\sphericalangle A$, there exists a line through D and not through A that intersects both sides of $\sphericalangle A$. COnclusion: The angle sum of every triangle is 180°.

Legendre's Axiom For any acute angle and any point in the interior of that angle, there exists a line through that point and not through the angle vertex that intersects both sides of the angle.

Material from Chapter 6 Negation of Hilbert's Euclidean Parallel Postulate There exist a line l and a point P not on l such that at least two lines parallel to l pass through P.

Basic Theorem 6.1 A non-Euclidean plane satisfying Aristotle's axiom satisfies the acute angle hypothesis. From the acute angle hypothesis alone, the following properties follow: The angle sum of every triangle is $< 180°$, the summit angles of all Saccheri quadrilaterals are acute, the fourth angle of every Lambert quadrilateral is acute and rectangles do not exist. The summit of a Saccheri quadrilateral is greater than the base. The segment joining the midpoints of the summit and the base is perpendicular to both, is the shortest segment between the base line and the summit line, and is the only common perpendicular segment between those lines. A side adjacent to the acute angle of a Lambert quadrilateral is greater than the opposite side.

Universal Non-Euclidean Theorem In a Hilbert plane in which rectangles do not exist, for every line l and every point P not on l, there are at least two parallels to l through P.

Corollary In a Hilbert plane in which rectangles do not exist, for every line l and every point P not on l, there are infinitely many parallels to l through P.

P6.1 (Additivity of Defect If D is any point between A and B then $\delta ABC = \delta ACD + \delta BCD$.

P6.2 (AAA) (No Similarity) In a plane satisfying the acute angle hypothesis, if two triangles are similar, then they are congruent.

P6.3 In a plane in which rectangles do not exist, if $l \parallel l'$, then any set of points on l equidistant from l' has at most two points in it.

P6.4 In a Hilbert plane satisfying the acute angle hypothesis, if $l \parallel l'$ and if there exists a pair of points A and B on l equidistant from l', then l and l' have a unique common perpendicular segment MM' dropped from the midpoint M of AB. MM' is the shortest segment joining a point of l to a point of l', and the segments AA' and BB' increase as A, B recede from M.

P6.5 In a Hilbert plane in which rectangles do not exist, if lines l and l' have a common perpendicular segment MM', then they are parallel and that common perpendicular segment is the unique. Moreover, if A and B are any points on l such that M is the midpoint of AB, then A and B are equidistant from l'.

Definition: Limiting Parallel Rays Given a line l and a point P not on l. Let Q be the foot of the perpendicular from P to l. A **limiting parallel ray** to l emanating from P is a ray $\overrightarrow{PX}$ that does not intersect l and such that for every ray $\overrightarrow{PY}$ which is between $\overrightarrow{PQ}$ and $\overrightarrow{PX}$, $\overrightarrow{PY}$ intersects l.

Advanced Theorem In non-Euclidean planes satisfying Aristotle's axiom and the line-circle continuity principle, limiting parallel rays exist for every line l and every point P not on l.

Hilbert's Hyperbolic Axiom of Parallels For every line l and every point P not on l, a limiting parallel ray $\overrightarrow{PX}$ emanating from P exists and it does **not** make a right angle with $\overrightarrow{PQ}$, where Q is the foot of the perpendicular from P to l.

Definition hyperbolic plane A Hilbert plane in which Hilbert's hyperbolic axiom of parallels holds is called a **hyperbolic plane**.

P6.6 In a hyperbolic plane, with notation as in the above definition, is acute. There is a ray $\overrightarrow{PX'}$, emanating from P, with X' on the opposite side of $\overleftrightarrow{PQ}$ from X, such that $\overrightarrow{PX'}$ is another limiting parallel ray to l and $\sphericalangle XPQ \cong \sphericalangle X'PQ$. These two rays, situated symmetrically about $\overrightarrow{PQ}$, are the only limiting parallel rays to l through P.

Definition: angles of parallelism With the above notation, acute angles $\sphericalangle XPQ$ and $\sphericalangle X'PQ$ are called **angles of parallelism** for segment PQ. Lobachevsky denoted any angle congruent to them by $\Pi(PQ)$.

Theorem 6.2 In a non-Euclidean plane satisfying Dedekind's axiom, Hilbert's hyperbolic axiom of parallels holds, as do Aristotle's axiom and the acute angle hypothesis.

Definition: Real Hyperbolic Plane A non-Euclidean plane satisfying Dedekind's axiom is called a **real hyperbolic plane**.

Corollary 1 All the results proved previously in Chapter 6 hold in real hyperbolic planes.

Corollary 2 A Hilbert plane satisfying Dedekind's axiom is either real Euclidean or real hyperbolic.

Theorem 6.3 In a hyperbolic plane, given m parallel to l such that m does not contain a limiting parallel ray to l in either direction. Then there exists a common perpendicular to m and l (unique by P6.5).

Theorem (Perpendicular Bisector Theorem Given any triangle in a hyperbolic plane, the perpendicular bisectors of its sides are concurrent in the projective completion.

HYPERBOLIC GEOMETRY

Results from chapter 7 (Contextual definitions not included):

Metamathematical Theorem 1 If Euclidean geometry is consistent, then so is hyperbolic geometry.

Corollary Proof is in text: If Euclidean geometry is consistent, then no proof of disproof of Euclid's parallel postulate from the axioms of neutral geometry will ever be found – Euclid's parallel postulate is independent of the other postulates.

P7.1
1. $P = P'$ if and only if P lies on the circle of inversion γ.
2. If P is inside γ then P' is outside γ, and if P is outside γ, then P' is inside γ.
3. $(P')' = P$.

P7.2 Suppose $P \neq O$ is inside γ. Let TU be the chord of γ which is perpendicular to $\overleftrightarrow{OP}$. Then the inverse P' of P is the pole of chord TU, i.e., the point of intersection of the tangents to γ at T and U.

P7.3 If P is outside γ, let Q be the midpoint of segment OP. Let σ be the circle with center Q and radius $\overline{OQ} = \overline{QP}$. Then σ cuts γ in two points T and U, $\overleftrightarrow{PT}$ and $\overleftrightarrow{PU}$ are tangent to γ, and the inverse P' of P is the intersection of TU and OP.

P7.4 Let T and U be points on γ that are not diametrically opposite and let P be the pole of TU. Then we have $PT \cong PU$, $\sphericalangle PTU \cong \sphericalangle PUT$, $\overleftrightarrow{OP} \perp \overleftrightarrow{TU}$, and the circle δ with center P and radius $\overline{PT} = \overline{PU}$ cuts γ orthogonally at T and U.

L7.1 Given that point O does not lie on circle δ.

1. If two lines through O intersect δ in pairs of points (P_1, P_2) and (Q_1, Q_2), respectively, then we have $(\overline{OP_1})(\overline{OP_2}) = (\overline{OQ_1})(\overline{OQ_2})$. This common product is called the *power* of O with respect to δ when O is outside of δ, and minus this number is called the power of O when O is inside δ.
2. If O is outside δ and a tangent to δ from O touches δ at point T, then $(\overline{OT})^2$ equals the power of O with respect to δ.

P7.5 Let P be any point which does not lie on circle γ and which does not coincide with the center O of γ, and let δ be a circle through P. Then δ cuts γ orthogonally if and only if δ passes through the inverse point P' of P with respect to γ.

Corollary Let P be as in Proposition 7.5. Then the locus of the centers of all circles δ through P orthogonal to γ is the line l, which is the perpendicular bisector of PP'. If P is inside γ, then line l is a line in the exterior of γ. Conversely, let l be any line in the exterior of γ, let C be the foot of the perpendicular from O to l, let δ be the circle centered at C which is orthogonal to δ (constructed as in Proposition 7.3), and let P be the intersection of δ with OC; then l is the locus of the centers of all circles orthogonal to γ that pass through P.

Definition: cross ratio Let A and B be points inside γ and let P and Q be the ends of the P-line through A and B. Define the **cross ratio** (AB, PQ) by $(AB, PQ) = \frac{(\overline{AP})(\overline{BQ})}{(\overline{BP})(\overline{AQ})}$.