Definitions, Axioms, Postulates, Propositions, and Theorems from *Euclidean and Non-Euclidean Geometries* by Marvin Jay Greenberg (2005-02-16)

Logic Rules (Greenberg): Logic Rule 1 Allowable justifications.

1. "By hypothesis ...".
2. "By axiom ...".
3. "By theorem ..." (previously proved).
4. "By definition ...".
5. "By step ..." (a previous step in the argument).
6. "By rule ..." of logic.

Logic Rule 2 Proof by contradiction (RAA argument).

Logic Rule 3 The tautology $\neg(\neg S) \iff S$

Logic Rule 4 The tautology $\neg(H \implies C) \iff H \wedge (\neg C)$.

Logic Rule 5 The tautology $\neg(S_1 \wedge S_2) \iff (\neg S_1 \vee \neg S_2)$.

Logic Rule 6 The statement $\neg(\forall x S(x))$ means the same as $\exists x(\neg S(x))$.

Logic Rule 7 The statement $\neg(\exists x S(x))$ means the same as $\forall x(\neg S(x))$.

Logic Rule 8 The tautology $((P \implies Q) \wedge P) \implies Q$.

Logic Rule 9 The tautologies

1. $((P \implies Q) \wedge (Q \implies R) \implies (P \implies R)$.
2. $(P \wedge Q) \implies P$ and $(P \wedge Q) \implies Q$.
3. $(\neg Q \implies \neg P) \implies ((P \implies Q)$.

Logic Rule 10 The tautology $P \implies (P \vee \neg P)$.

Logic Rule 11 (Proof by Cases) If C can be deduced from each of $S_1, S_2, \cdots, S_n$ **individually**, then $(S_1 \vee S_2 \vee \cdots S_n) \implies C$ is a tautology.

Undefined Terms: Point, Line, Incident, Between, Congruent.

Incidence Axioms:

IA1: For every two distinct points there exists a unique line incident on them.

IA2: For every line there exist at least two points incident on it.

IA3: There exist three distinct points such that no line is incident on all three.

Incidence Propositions:

P2.1: If l and m are distinct lines that are non-parallel, then l and m have a unique point in common.

P2.2: There exist three distinct lines such that no point lies on all three.

P2.3: For every line there is at least one point not lying on it.

P2.4: For every point there is at least one line not passing through it.

P2.5: For every point there exist at least two distinct lines that pass through it.

Betweenness Axioms:

B1: If $A * B * C$, then A, B, and C are three distinct points all lying on the same line, and $C * B * A$.

B2: Given any two distinct points B and D, there exist points A, C, and E lying on $\overleftrightarrow{BD}$ such that $A*B*D$, $B*C*D$, and $B*D*E$.

B3: If A, B, and C are three distinct points lying on the same line, then one and only one of them is between the other two.

B4: For every line l and for any three points A, B, and C not lying on l:

1. If A and B are on the same side of l, and B and C are on the same side of l, then A and C are on the same side of l.
2. If A and B are on opposite sides of l, and B and C are on opposite sides of l, then A and C are on the same side of l.

Corollary If A and B are on opposite sides of l, and B and C are on the same side of l, then A and C are on opposite sides of l.

Betweenness Definitions:

Segment AB: Point A, point B, and all points P such that $A*P*B$.

Ray $\overrightarrow{AB}$: Segment AB and all points C such that $A*B*C$.

Line $\overleftrightarrow{AB}$: Ray $\overrightarrow{AB}$ and all points D such that $D*A*B$.

Same/Opposite Side: Let l be any line, A and B any points that do not lie on l. If $A = B$ or if segment AB contains no point lying on l, we say A and B are *on the same side of* l, whereas if $A \neq B$ and segment AB does intersect l, we say that A and B are *on opposite sides of* l. The law of excluded middle tells us that A and B are either on the same side or on opposite sides of l.

Betweenness Propositions:

P3.1 (does not use BA-4): For any two points A and B:

1. **Proof in 3rd Ed.**$\overrightarrow{AB} \cap \overrightarrow{BA} = AB$, and
2. $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$.

P3.2 Proof in 3rd Ed.: Every line bounds exactly two half-planes and these half-planes have no point in common.

Same Side Lemma: Given $A*B*C$ and l any line other than line $\overleftrightarrow{AB}$ meeting line $\overleftrightarrow{AB}$ at point A, then B and C are on the same side of line l.

Opposite Side Lemma: Given $A*B*C$ and l any line other than line $\overleftrightarrow{AB}$ meeting line $\overleftrightarrow{AB}$ at point B, then A and C are on opposite sides of line l.

P3.3 Proof in 3rd Ed.: Given $A*B*C$ and $A*C*D$. Then $B*C*D$ and $A*B*D$.

P3.4 Proof in 3rd Ed.: If $C*A*B$ and l is the line through A, B, and C, then for every point P lying on l, P either lies on ray $\overrightarrow{AB}$ or on the opposite ray $\overrightarrow{AC}$.

Pasch's Theorem Proof in 3rd Ed.: If A, B, and C are distinct points and l is any line intersecting AB in a point between A and B, then l also intersects either AC, or BC. If C does not lie on l, then l does not intersect both AC and BC.

P3.5: Given $A*B*C$. Then $AC = AB \cup BC$ and B is the only point common to segments AB and BC.

P3.6: Given $A*B*C$. Then B is the only point common to rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$, and $\overrightarrow{AB} = \overrightarrow{AC}$.

Angle Definitions:

Interior: Given an angle $\sphericalangle CAB$, define a point D to be in the *interior* of $\sphericalangle CAB$ if D is on the same side of $\overleftrightarrow{AC}$ as B and if D is also on the same side of $\overleftrightarrow{AB}$ as C. Thus, the interior of an angle is the intersection of two half-planes. (Note: the interior does not include the angle itself, and points not on the angle and not in the interior are on the exterior).

Ray Betweenness: Ray $\overrightarrow{AD}$ is *between* rays $\overrightarrow{AC}$ and $\overrightarrow{AB}$ provided $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not opposite rays and D is interior to $\sphericalangle CAB$.

Interior of a Triangle: The interior of a triangle is the intersection of the interiors of its thee angles. Define a point to be *exterior* to the triangle if it in not in the interior and does not lie on any side of the triangle.

Triangle: The union of the three segments formed by three non-collinear points.

Angle Propositions:

P3.7: Given an angle $\sphericalangle CAB$ and point D lying on line $\overleftrightarrow{BC}$. Then D is in the interior of $\sphericalangle CAB$ iff $B{*}D{*}C$.

"Problem 9": Given a line l, a point A on l and a point B not on l. Then every point of the ray $\overrightarrow{AB}$ (except A) is on the same side of l as B.

P3.8: If D is in the interior of $\sphericalangle CAB$, then:

1. so is every other point on ray $\overrightarrow{AD}$ except A,
2. no point on the opposite ray to $\overrightarrow{AD}$ is in the interior of $\sphericalangle CAB$, and
3. if $C{*}A{*}E$, then B is in the interior of $\sphericalangle DAE$.

Crossbar Theorem: If $\overrightarrow{AD}$ is between $\overrightarrow{AC}$ and $\overrightarrow{AB}$, then $\overrightarrow{AD}$ intersects segment BC.

P3.9:

1. If a ray r emanating from an exterior point of $\triangle ABC$ intersects side AB in a point between A and B, then r also intersects side AC or BC.
2. If a ray emanates from an interior point of $\triangle ABC$, then it intersects one of the sides, and if it does not pass through a vertex, then it intersects only one side.

Congruence Axioms:

C1: If A and B are distinct points and if A' is any point, then for each ray r emanating from A' there is a *unique* point B' on r such that $B' \neq A'$ and $AB \cong A'B'$.

C2: If $AB \cong CD$ and $AB \cong EF$, then $CD \cong EF$. Moreover, every segment is congruent to itself.

C3: If $A{*}B{*}C$, and $A'{*}B'{*}C'$, $AB \cong A'B'$, and $BC \cong B'C'$, then $AC \cong A'C'$.

C4: Given any $\sphericalangle BAC$ (where by definition of angle, $\overrightarrow{AB}$ is not opposite to $\overrightarrow{AC}$ and is distinct from $\overrightarrow{AC}$), and given any ray $\overrightarrow{A'B'}$ emanating from a point A', then there is a *unique* ray $\overrightarrow{A'C'}$ on a given side of line $\overleftrightarrow{A'B'}$ such that $\sphericalangle B'A'C' \cong \sphericalangle BAC$.

C5: If $\sphericalangle A \cong \sphericalangle B$ and $\sphericalangle A \cong \sphericalangle C$, then $\sphericalangle B \cong \sphericalangle C$. Moreover, every angle is congruent to itself.

C6 (SAS): If two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of another triangle, then the two triangles are congruent.

Congruence Propositions:

Corollary to SAS: Proof in 3rd Ed. Given $\triangle ABC$ and segment $DE \cong AB$, there is a unique point F on a given side of line $\overleftrightarrow{DE}$ such that $\triangle ABC \cong \triangle DEF$.

P3.10 Proof in 3rd Ed.: If in $\triangle ABC$ we have $AB \cong AC$, then $\sphericalangle B \cong \sphericalangle C$.

P3.11: If $A{*}B{*}C$, $D{*}E{*}F$, $AB \cong DE$, and $AC \cong DF$, then $BC \cong EF$.

P3.12 Proof in 3rd Ed.: Given $AC \cong DF$, then for any point B between A and C, there is a unique point E between D and F such that $AB \cong DE$.

P3.13:
1. Exactly one of the following holds: $AB < CD$, $AB \cong CD$, or $AB > CD$.
2. If $AB < CD$ and $CD \cong EF$, then $AB < EF$.
3. If $AB > CD$ and $CD \cong EF$, then $AB > EF$.
4. If $AB < CD$ and $CD < EF$, then $AB < EF$.

P3.14: Supplements of Congruent angles are congruent.

P3.15:
1. Vertical angles are congruent to each other.
2. An angle congruent to a right angle is a right angle.

P3.16 Proof in 3rd Ed.: For every line l and every point P there exists a line through P perpendicular to l.

P3.17 (ASA): Given $\triangle ABC$ and $\triangle DEF$ with $\sphericalangle A \cong \sphericalangle D$, $\sphericalangle C \cong \sphericalangle F$, and $AC \cong DF$, then $\triangle ABC \cong \triangle DEF$.

P3.18: In in $\triangle ABC$ we have $\sphericalangle B \cong \sphericalangle C$, then $AB \cong AC$ and $\triangle ABC$ is isosceles.

P3.19 Proof in 3rd Ed.: Given $\overrightarrow{BG}$ between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, $\overrightarrow{EH}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$, $\sphericalangle CBG \cong \sphericalangle FEH$ and $\sphericalangle GBA \cong \sphericalangle HED$. Then $\sphericalangle ABC \cong \sphericalangle DEF$.

P3.20: Given $\overrightarrow{BG}$ between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, $\overrightarrow{EH}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$, $\sphericalangle CBG \cong \sphericalangle FEH$ and $\sphericalangle ABC \cong \sphericalangle DEF$. Then $\sphericalangle GBA \cong \sphericalangle HED$.

P3.21:
1. Exactly one of the following holds: $\sphericalangle P < \sphericalangle Q$, $\sphericalangle P \cong \sphericalangle Q$, or $\sphericalangle P > \sphericalangle Q$.
2. If $\sphericalangle P < \sphericalangle Q$ and $\sphericalangle Q \cong \sphericalangle R$, then $\sphericalangle P < \sphericalangle R$.
3. If $\sphericalangle P > \sphericalangle Q$ and $\sphericalangle Q \cong \sphericalangle R$, then $\sphericalangle P > \sphericalangle R$.
4. If $\sphericalangle P < \sphericalangle Q$ and $\sphericalangle Q < \sphericalangle R$, then $\sphericalangle P < \sphericalangle R$.

P3.22 (SSS): Given $\triangle ABC$ and $\triangle DEF$. If $AB \cong DE$, $BC \cong EF$, and $AC \cong DF$, then $\triangle ABC \cong \triangle DEF$.

P3.23 Proof in 3rd Ed.: All right angles are congruent to each other.

Corollary (not numbered in text) If P lies on l then the perpendicular to l through P is unique.

Definitions:

Segment Inequality: $AB < CD$ (or $CD > AB$) means that there exists a point E between C and D such that $AB \cong CE$.

Angle Inequality: $\sphericalangle ABC < \sphericalangle DEF$ means there is a ray $\overrightarrow{EG}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$ such that $\sphericalangle ABC \cong \sphericalangle GEF$.

Right Angle: An angle $\sphericalangle ABC$ is a right angle if has a supplementary angle to which it is congruent.

Parallel: Two lines l and m are parallel if they do not intersect, i.e., if no point lies on both of them.

Perpendicular: Two lines l and m are perpendicular if they intersect at a point A and if there is a ray $\overrightarrow{AB}$ that is a part of l and a ray $\overrightarrow{AC}$ that is a part of m such that $\sphericalangle BAC$ is a right angle.

Triangle Congruence and Similarity: Two triangles are congruent if a one-to-one correspondence can be set up between their vertices so that corresponding sides are congruent and corresponding angles are congruent. Similar triangles have this one-to-one correspondence only with their angles.

Circle (with center O and radius OA): The set of all points P such that OP is congruent to OA.

Triangle: The set of three distinct segments defined by three non-collinear points.

Continuity Axioms:

Archimedes' Axiom: If AB and CD are any segments, then there is a number n such that if segment CD is laid off n times on the ray $\overrightarrow{AB}$ emanating from A, then a point E is reached where $n \cdot CD \cong AE$ and B is between A and E.

Dedekind's Axiom: Suppose that the set of all points on a line l is the union $\Sigma_1 \cup \Sigma_2$ of two nonempty subsets such that no point of Σ_1 is between two points of Σ_2 and visa versa. Then there is a unique point O lying on l such that $P_1 * O * P_2$ if and only if one of P_1, P_2 is in Σ_1, the other in Σ_2 and $O \neq P_1, P_2$. A pair of subsets Σ_1 and Σ_2 with the properties in this axiom is called a Dedekind cut of the line l.

Continuity Principles: Circular Continuity Principle: If a circle γ has one point inside and one point outside another circle γ', then the two circles intersect in two points.

Elementary Continuity Principle: In one endpoint of a segment is inside a circle and the other outside, then the segment intersects the circle.

Other Theorems, Propositions, and Corollaries in Neutral Geometry:

T4.1 Proof in 3rd Ed.: If two lines cut by a transversal have a pair of congruent alternate interior angles, then the two lines are parallel.

Corollary 1 Proof in 3rd Ed.: Two lines perpendicular to the same line are parallel. Hence the perpendicular dropped from a point P not on line l to l is unique.

Corollary 2 Proof in 3rd Ed.: If l is any line and P is any point not on l, there exists at least one line m through P parallel to l.

T4.2 (Exterior Angle Theorem) Proof in 3rd Ed.: An exterior angle of a triangle is greater than either remote interior angle.

P4.1 (SAA): Given $AC \cong DF$, $\sphericalangle A \cong \sphericalangle D$, and $\sphericalangle B \cong \sphericalangle E$. Then $\triangle ABC \cong \triangle DEF$.

P4.2: Two right triangles are congruent if the hypotenuse and leg of one are congruent respectively to the hypotenuse and a leg of the other.

P4.3: Every segment has a unique midpoint.

P4.4:

1. Every angle has a unique bisector.
2. Every segment has a unique perpendicular bisector.

P4.5: In a triangle $\triangle ABC$, the greater angle lies opposite the greater side and the greater side lies opposite the greater angle, i.e., $AB > BC$ if and only if $\sphericalangle C > \sphericalangle A$.

P4.6: Given $\triangle ABC$ and $\triangle A'B'C'$, if $AB \cong A'B'$ and $BC \cong B'C'$, then $\sphericalangle B < \sphericalangle B'$ if and only if $AC < A'C'$.

T4.3 (see text for details): There is a unique way of assigning a degree measure to each angle, and, given a segment OI, called a unit segment, there is a unique way of assigning a length to each segment AB that satisfy our standard uses of angle and length.

Corollary 1 Proof in 3rd Ed.: The sum of the degree measures of any two angles of a triangle is less than 180°.

Corollary 2 Proof in 3rd Ed.: If A, B, and C are three noncollinear points, then $\overline{AC} < \overline{AB} + \overline{BC}$.

T4.4 (Saccheri-Legendre) Proof in 3rd Ed.: The sum of the degree measures of the three angles in any triangle is less than or equal to 180°.

Corollary 1 Proof in 3rd Ed.: The sum of the degree measures of two angles in a triangle is less than or equal to the degree measure of their remote exterior angle.

Corollary 2 Proof in 3rd Ed.: The sum of the degree measures of the angles in any convex quadrilateral is at most 360° (**note: quadrilateral $\square ABCD$ is convex if it has a pair of opposite sides such that each is contained in a half-plane bounded by the other.**)

Note: Statements up to this point are from or form neutral geometry. Choosing Hilbert's/Euclid's Axiom (the two are logically equivalent) or the Hyperbolic Axiom will make the geometry Euclidean or Hyperbolic, respectively.

Parallelism Axioms:

Hilbert's Parallelism Axiom for Euclidean Geometry: For every line l and every point P not lying on l there is at most one line m through P such that m is parallel to l. (Note: it can be proved from the previous axioms that, assuming this axiom, there is **EXACTLY** one line m parallel to l [see T4.1 Corollary 2]).

Euclid's Fifth Postulate: If two lines are intersected by a transversal in such a way that the sum of the degree measures of the two interior angles on one side of the transversal is less than 180°, then the two lines meet on that side of the transversal.

Hyperbolic Parallel Axiom: There exist a line l and a point P not on l such that at least two distinct lines parallel to l pass through P.

Equivalences to Hilbert's Parallel Postulate (HPP):

T4.5 Proof in 3rd Ed.: Euclid's Fifth Postulate $\Longleftrightarrow$ HPP.

P4.7: If a line intersects one of two parallel lines, then it also intersects the other $\Longleftrightarrow$ HPP.

P4.8: Converse to Theorem 4.1 $\Longleftrightarrow$ HPP.

P4.9: If t is transversal to l and m, $l \| m$, and $t \perp l$, then $t \perp m \Longleftrightarrow$ HPP.

P4.10: If $k \| l$, $m \perp k$, and $n \perp l$, then either $m = n$ or $m \| n \Longleftrightarrow$ HPP.

P4.11: The angle sum of every triangle is 180° $\Longleftrightarrow$ HPP.

Definitions: Angle sum of a triangle: The angle sum of triangle $\triangle ABC$ is the sum of the degree measures of the three angles of the triangle.

Defect of a triangle: The defect, $\delta(ABC)$, of triangle $\triangle ABC$ is 180° minus the angle sum.

Theorems 4.6 and 4.7 (see text) are used to prove the converse of P4.11. They define the *defect* of a triangle to be the 180° minus the angle sum, then show that if one defective triangle exists, then all triangles are defective. Or, in contrapositive form, if one triangle has angle sum 180°, then so do all others. They do not assume a parallel postulate.

Results about defect T4.6 Proof in 3rd Ed.: Let $\triangle ABC$ be any triangle and D a point between A and B. Then $\delta(ABC) = \delta(ACD) + \delta(BCD)$.

Corollary Proof in 3rd Ed.: Under the same hypotheses, the angle sum of $\triangle ABC$ is equal to 180° if and only if the angle sums of both $\triangle ACD$ and $\triangle BCD$ are equal to 180°.

T4.7 Proof in 3rd Ed.: If a triangle existis whose angle sum is 180°, then a rectangle exists. If a rectangle exists then every triangle has angle sum equal to 180°.

Corollary: If there exists a triangle with positive defect then all triangles have positive defect.

Results from Chapter 5 Wallis: Given any triangle $\triangle ABC$ and given any segment DE. The existance of a triangle $\triangle DEF$ (having DE as one of its sides) that is similar to $\triangle ABC$ (denoted $\triangle DEF \sim \triangle ABC$) $\Longleftrightarrow$ HPP.

Claraut's Axiom: Rectangles exist $\Longleftrightarrow$ HPP.

T5.1 Proof in 3rd Ed.: If for any acute angle and any point D in the interior of the angle, there exists a line through D and not through the vertex of the angle which intersects both sides of the angle, then the angle sum of every triangle is 180°.

Theorems Using the Parallel Axiom

Parallel Projection Theorem: Given three parallel lines l, m, and n. Let t and t' be transversals to these parallels, cutting them in points A, B, and C and in points A', B', and C', respectively. Then $\overline{AB}/\overline{BC} = \overline{A'B'}/\overline{B'C'}$.

Fundamental Theorem on Similar Triangles: Given $\triangle ABC \sim \triangle A'B'C'$. Then the corresponding sides are proportional.

HYPERBOLIC GEOMETRY

L6.1: In hyperbolic geometry rectangles do not exist.

Universal Hyperbolic Theorem Proof in 3rd Ed.: In hyperbolic geometry, from every line l and every point P not on l there pass through P at least two distinct parallels to l.

Corollary Proof in 3rd Ed.: In hyperbolic geometry, for every line l and every point P not on l, there are infinitely many parallels to l through P.

T6.1: In hyperbolic geometry, all triangles have angle sum less than 180°.

Corollary: In hyperbolic geometry, all convex quadrilaterals have angle sum less than 360°.

T6.2 Proof in 3rd Ed.: In hyperbolic geometry if two triangles are similar, they are congruent.

T6.3 Proof in 3rd Ed.: In hyperbolic geometry if l and l' are any distinct parallel lines, then any set of points on l equidistant from l' has at most two points in it.

T6.4 Proof in 3rd Ed.: In hyperbolic geometry if l and l' are parallel lines for which there exists a pair of points A and B on l equidistant from l', then l and l' have a common perpendicular segment that is also the shortest segment between l and l'.

L6.2 Proof in 3rd Ed.: The segment joining the midpoints of the base and summit of a Saccheri quadrilateral is perpendicular to both the base and the summit, and this segment is shorter than the sides.

T6.5 Proof in 3rd Ed.: If lines l and l' have a common perpendicular MM', then they are parallel and MM' is unique. Moreover, if A and B are points on l such that M is the midpoint of segment AB, then A and B are equidistant from l'.

T6.6 Proof in 3rd Ed.: For every line l and every point P not on l, let Q be the foot of the perpendicular from P to l. Then there are two unique rays $\overrightarrow{PX}$ and $\overrightarrow{PX'}$ on opposite sides of $\overleftrightarrow{PQ}$ that do not meet l and have the property that a ray emanating from P meets l if and only if it is between $\overrightarrow{PX}$ and $\overrightarrow{PX'}$. Moreover, these limiting rays are situated symmetrically about $\overleftrightarrow{PQ}$ in the sense that $\measuredangle XPQ \cong \measuredangle X'PQ$.

T6.7 Proof in 3rd Ed.: Given m parallel to l such that m does not contain a limiting parallel ray to l in either direction. Then there exists a common perpendicular to m and l, which is unique.

Results from chapter 7 (Contextual definitions not included):

Metamathematical Theorem 1 If Euclidean geometry is consistent, then so is hyperbolic geometry.

Corollary Proof in 3rd Ed.: If Euclidean geometry is consistent, then no proof of disproof of the parallel postulate from the rest of Hilbert's postulates will ever be found. That is, the parallel postulate is independent of the other postulates of Euclidean geometry.

P7.1
1. $P = P'$ if and only if P lies on the circle of inversion γ.
2. If P is inside γ then P' is outside γ, and if P is outside γ, then P' is inside γ.
3. $(P')' = P$.

P7.2 Suppose P is inside γ. Let TU be the chord of γ which is perpendicular to $\overleftrightarrow{OP}$. Then the inverse P' of P is the pole of chord TU, i.e., the point of intersection of the tangents to γ at T and U.

P7.3 If P is outside γ, let Q be the midpoint of segment OP. Let σ be the circle with center Q and radius $\overline{OQ} = \overline{QP}$. Then σ cuts γ in two points T and U, $\overleftrightarrow{PT}$ and $\overleftrightarrow{PU}$ are tangent to γ, and the inverse P' of P is the intersection of TU and OP.

P7.4 Let T and U be points on γ that are not diametrically opposite and let P be the pole of TU. Then $PT \cong PU$, $\measuredangle PTU \cong \measuredangle PUT$, $\overleftrightarrow{OP} \perp \overleftrightarrow{TU}$, and the circle δ with center P and radius $\overline{PT} = \overline{PU}$ cuts γ orthogonally at T and U.

L7.1 Given that point O does not lie on circle δ.
1. If two lines through O intersect δ in pairs of points (P_1, P_2) and (Q_1, Q_2), respectively, then we have $(\overline{OP_1})(\overline{OP_2}) = (\overline{OQ_1})(\overline{OQ_2})$. This common product is called the *power* of O with respect to δ when O is outside of δ, and minus this number is called the power of O when O is inside δ.
2. If O is outside δ and a tangent to δ from O touches δ at point T, then $(\overline{OT})^2$ equals the power of O with respect to δ.

P7.5 Let P be any point which does not lie on circle γ and which does not coincide with the center O of γ, and let δ be a circle through P. Then δ cuts γ orthogonally if and only if δ passes through the inverse point P' of P with respect to γ.