Mathematics 434-A Spring 2001 Problem Sheet 1

February 6, 2001

Exercises

1. Let $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Find an orthonormal basis for R^2 with respect to the form $\langle X, Y \rangle = X^t AY$.

2. Prove any positive definite form is nondegenerate.

3. Find an othogonal basis for the form on R^3 whose matrix (with respect to the standard basis) is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

4. Let A be the matrix of a symmetric bilinear form $\langle , \rangle$ with respect to some basis. Prove or disprove: The eigenvalues of A are independent of the basis. That is, if A' is the matrix of the form with respect to another basis, then A and A' have the same eigenvalues.

5. Let W be a subspace of a vector space V on which a symmetric bilinear form is given.

   (a) Prove that $W^\perp$ is a subspace of V.

   (b) Prove that the null space N is a subspace of V.

6. Let W_1, W_2 be subspaces of a vector space V with a symmetric bilinear form. Prove:

   (a) $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$

   (b) $W \subset \left(W^\perp\right)^\perp$

   (c) If $W_1 \subset W_2$ then $W_2^\perp \subset W_1^\perp$.

0.1 Examples in class:

1. Let V denote the vector space of all real $n \times n$ matrices. Prove that $\langle A, B \rangle = Trace\left(A^t B\right)$ is a positive definite, symmetric bilinear form on V. Find an orthonormal basis for this form.

   (a) $\langle A, B \rangle = Trace\left(A^t B\right) = Trace\left(\left(A^t B\right)^t\right) = Trace\left(B^t A\right) = \langle B, A \rangle$

   (b) $\langle A, A \rangle = Trace\left(A^t A\right) = \|v_1\|^2 + \cdots + \|v_n\|^2$ where v_i denotes the ith column of A.

   (c) $\langle e_{ij}, e_{rs} \rangle = Trace\left(e_{ji} e_{rs}\right) = \begin{cases} 1, & i = r \text{ and } j = s \\ 0, & \text{otherwise} \end{cases}$ Thus, the standard basis is orthonormal with respect to $\langle , \rangle$.

2. Let V be the vector space of all real 2×2 matrices.

   (a) Show the form $\langle A, B \rangle = \det(A + B) - \det(A) - \det(B)$ is symmetric and bilinear.

      i. $\langle A, B \rangle = \det(A + B) - \det(A) - \det(B) = \det(B + A) - \det(B) - \det(A) = \langle B, A \rangle$

ii. $\langle A+B, C\rangle = \det(A+B+C) - \det(A+B) - \det(C)$; $\langle A, C\rangle + \langle B, C\rangle = \det(A+C) - \det(A) - \det(C) + \det(B+C) - \det(B) - \det(C)$ and push the 2×2 size.

(b) Compute the matrix of this form with respect to the standard basis and determine the signature of the form.

i. $\langle e_{ij} e_{rs}\rangle = \det(e_{ij}+e_{rs}) - \det(e_{ij}) - \det(e_{rs}) = \begin{cases} 1, & i=j=1 \text{ and } r=s=2 \\ -1, & ij=12 \text{ and } rs=21 \\ 0, & \text{otherwise} \end{cases}$

ii. $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ Permute to $A' = \begin{bmatrix} I_2 & \\ & -I_2 \end{bmatrix}$.

iii. Signature is $2+2=4$.

(c) Do the same for the subspace of matrices with trace zero.

i. Basis $\{e_{11}, e_{12}, e_{21}\}$ $e_{22} = -e_{11}$.