1   Mathematics 433   Fall 2000   Problems to Turn in: 5

October 17, 2001

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use the results of discussion, a text, notes, or technology. **Only write on one side of each page.**

"To those who do not know mathematics it is difficult to get across a real feeling as to the beauty, the deepest beauty of nature. If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speaks in". -Richard Feynman (1918-1988)

"By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems, and, in effect, increases the mental power of the race." – Alfred North Whitehead

Problems

1. Do the following.

   (a) Prove that the relation x is conjugate to y in a group G is an equivalence relation on G.

   (b) Describe the elements a whose conjugacy class (that is, whose equivalence class) consists of the element a alone.

2. Classify groups of order 6 by analyzing the following three cases.

   (a) G contains an element of order 6.

   (b) G contains an element of order 3 but none of order 6.

   (c) All elements of G have order 1 or 2.

3. Prove the *Correspondence Theorem.*

   Theorem 1 *Let $\phi : G \to G'$ be an onto homomorphism between groups G and G' and with $\ker(\phi) = N$. Show the set of subgroups of G', $S = \{H' : H' \leq G'\}$ is in one-to-one correspondence with the set $T = \{H : H \leq G \text{ and } N \subset H\}$ of all subgroups of G that contain $H = \ker(\phi)$. I suggest you use the map $\lambda : S \to T$ where λ takes the subgroup H of G to the subgroup $\phi(H)$ of G'. That is, $\lambda(H) = \phi(H)$. Also prove that if H is a normal subgroup of G then $\lambda(H)$ is a normal subgroup of G'.*

   [It might be useful to explicitly work out the correspondence above in the special case when G is a cyclic group of order 12 generated by x, G' is a cyclic group of order 6 generated by y and ϕ is the map given by $\phi(x^i) = y^i$.

4. Do both of the following.

   (a) Prove the cartesian product of two infinite cyclic groups is not infinite cyclic.

   (b) Prove the center of the cartesian product of two groups is the cartesian product of their centers.

5. Do both of the following.

   (a) Prove every integer a is congruent to the sum of its digits modulo 9.

   (b) Prove the associative and commutative laws for multiplication in $\mathbf{Z}/n\mathbf{Z}$.

6. Prove the subset $G \times \{e'\}$ of the product group $G \times G'$ is a normal subgroup isomorphic to G. Also prove that
$$\frac{G \times G'}{G \times \{e'\}} \approx G'.$$

7. Let G be a finite group of rotations in the plane about the origin. Prove G is cyclic.

8. Prove a linear operator on $\mathbf{R}^2$ is a reflection if and only if its eigenvalues are 1 and -1 and its eigenvectors are orthogonal.