Mathematics 433 — Fall 2001

Final Exam

December 11, 2001

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Name

Textbook/Notes used: ____

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use a text or notes or technology. **Only write on one side of each page.**

The Problems

Do any seven (7) of the following nine (9) problems.

1. Prove that a group of order 595 cannot be simple.

2. Do **one** of the following.

   (a) Show that no group G of size 224 can be simple. You may not use the Index or Embedding or Generalized Cayley Theorems. [Hint: see the useful information below.]

   (b) Show that there is no simple group of order pqr where p, q, r are primes (p, q, and r need not be distinct).

3. Do **one** of the following.

   (a) Let G be a finite group and H a normal Sylow p - subgroup of G. Show that $\phi(H) = H$ for every automorphism ϕ of G.

   (b) Show the center of a group of order 60 cannot have order 4.

4. Do **one** of the following.

   (a) Write out the homotopy that shows the fundamental group satisfies the associative property. The appropriate picture is on the blackboard.

   (b) Prove that a group of order 595 must have a normal Sylow 17 - subgroup.

5. Prove the **commutator subgroup** G' is indeed a subgroup of G.

   Definition (a) Let G be a group. The **commutator subgroup** G' of G is the subgroup generated by the set $\{aba^{-1}b^{-1} : a, b \in G\}$ .That is, every element of G' is a product

   $$x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}$$

   where each $x_i$has the form $aba^{-1}b^{-1}$, each $i_j = \pm 1$, and k is any positive integer.

   If $a, b \in G$ then we define the **commutator** of a and b to be $[a, b] = aba^{-1}b^{-1}$. (Note that a and b commute if and only if their commutator $[a, b] = e$.)

6. Do **one** of the following:

(a) Prove that the commutator subgroup G' of a group G is a normal subgroup of G. [Hint: $\left(g\left(aba^{-1}b^{-1}\right)g^{-1}\right)\left(g\left(cdc^{-1}d^{-1}\right)g^{-1}\right) = g\left(aba^{-1}b^{-1}\right)\left(cdc^{-1}d^{-1}\right)g^{-1}$.]

(b) If M and N are normal subgroups of G, $M \cap N = \{e\}$ and $a \in M$, $b \in N$ show that $ab = ba$.

7. Do **one** of the following.

   (a) Let a belong to a group G where $|a|$ is finite and let ϕ_a be the automorphism of G given by $\phi_a(x) = axa^{-1}$. Show that $|\phi_a|$ divides $|a|$.

   (b) Let Q denote the group of rational numbers under **addition**. Let $\phi : Q \to Q$ be an arbitrary automorphism and suppose $\phi(1) = a$.

      i. Prove $\phi(1/2) = a/2$.

      ii. Generalize the above to deduce $\phi(x) = x\phi(1)$ for all $x \in Q$.

8. Do **one** of the following.

   (a) Let S be the set of subsets of order 2 of the dihedral group D_3.Determine the orbits for the action of D_3 on S by conjugation.

   (b) Determine the class equation of the dihedral group D_{11}.

9. Do **one** of the following.

   (a) What is the stabilizer of the coset aH for the action of G on $G/H = \{xH : x \in G\}$ where the action is left multiplication.

   (b) Consider the operation of left multiplication by G on the set of its subsets. Let U be a subset whose orbit $\{gU : g \in G\}$ partitions G. Let H be the unique subset in this orbit which contains the identity e of G. Prove H is a subgroup of G and the sets gU are the left cosets of H.

Useful Information

1. Let $H_1, \cdots, H_k$ be a complete list of all p - Sylow subgroups of a finite group G.

Fact 1. For any $g \in G$ we have $\cap_{i=1}^{k}\left(gH_ig^{-1}\right) = \cap_{i=1}^{k} H_i$

   i. [This is easy to see since$\{H_1, \cdots, H_k\} = \{gH_1g^{-1}, \cdots, gH_kg^{-1}\}$ follows from $gH_ig^{-1} = gH_jg^{-1}$ if and only if $H_i = H_j$.]

Fact 2. The intersection $\cap_{i=1}^{k} H_i$ is a normal subgroup of G.

   i. [This follows since

$$\begin{aligned} x &\in g\left(\cap_{i=1}^{k} H_i\right)g^{-1} \\ &\Leftrightarrow g^{-1}xg \in H_i,\ i = 1, \cdots, k \\ &\Leftrightarrow x \in gH_ig^{-1},\ \ i = 1, \cdots, k \\ &\Leftrightarrow x \in \cap_{i=1}^{k}\left(gH_ig^{-1}\right) \end{aligned}$$

   and the last set is $\cap_{i=1}^{k} H_i$ by Fact 1.