Mathematics 433-A Fall 2001

Exam 1

October 2, 2000

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Name

Technology used: ____

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use a text or notes or technology. **Only write on one side of each page.**

The Problems

1. (20 points) Use mathematical induction to solve **one** of the following.

   (a) Let $\phi : G \to G'$ be a group homomorphism. Prove that for any elements $a_1, \cdots, a_k$ of G, $\phi(a_1 \cdots a_k) = \phi(a_1) \cdots \phi(a_k)$.

   (b) Compute the determinant of the $n \times n$ matrix A_n given by $A_n = \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdots & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix}$

2. (20 points) Given a group G, subgroup H of G and element $g \in G$,we define the **conjugate subgroup of** H in G to be the set

   $$gHg^{-1} = \left\{ ghg^{-1} : h \in H \right\}.$$

   Prove gHg^{-1} is indeed a subgroup of G.

3. (15 points) Let $\phi : G \to G'$ be an onto homomorphism and let N be a normal subgroup of G. Prove that $\phi(N)$ is a normal subgroup of G'.

4. (15 points each) Do any **three** (3) of the following problems.

   (a) Prove that a group of order 30 can have at most 7 subgroups of order 5.

   (b) Let a, b be elements in a group G.

      i. Suppose the product ab has finite order in G. Prove the orders, in G, of ab and ba are the same.

      ii. Must the orders of ab and ba be the same if the product ab has infinite order in G?

   (c) Let $\phi : G \to G'$ be a group homomorphism with kernel K.Let H be another subgroup of G. Recall that $HK = \{hk : \ h \in H, \ k \in K\}$ .Show $\phi^{-1}(\phi(H)) = HK$.

   (d) Let G, G' be groups.

      i. What is the order of the product group $G \times G'$?

      ii. Let $x \in G$ have order m and $y \in G'$ have order n. What is the order of $(x, y) \in G \times G'$? Prove your answer is correct.

   (e) Let S be a set of groups. Define the relation ˜ on S by If G˜H if and only if G is isomorphic to H. Show this relation is an equivalence relation on S.

   (f) Let Here $\psi : G \to H$ and $\phi : H \to K$ be homomorphisms. Thus, the composition $\phi \circ \psi : G \to K$ is a function. Prove $\phi \circ \psi$ is also a homomorphism. Describe the kernel of $\phi \circ \psi$.