1 Mathematics 433-A Fall 2000

1.1 Exam 2

November 21, 2000

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Name

Technology used: ______________________________________

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use a text or notes or technology. Include a careful sketch of any graph obtained by technology in solving a problem. **Only write on one side of each page.**

The Problems

1. (6 points each) Give the definitions of the following.
   (a) The **product group** $G \times G'$ of two groups G and G'.
   (b) The **quotient group** G/K of a group G by a normal subgroup K. Be sure to indicate the binaray operation in G/K.
   (c) The **orbit** of an element $s \in S$ where G is a group acting on the set S.
   (d) The **stabilizer** of an element $s \in S$ where G is a group acting on the set S.
   (e) A **rigid motion** of the plane to itself.

2. (10 points each) If G is a group acting on the set S, the element s is arbitrary in S, and $G_s = H$ is the stabilizer of s in G, then there is a map from the coset space of G_s in G to the orbit of s defined by

$$\begin{aligned} \phi &: G/G_s \to O_s \\ \phi(aH) &= as \end{aligned}$$

   Prove that this map ϕ is

   (a) one-to-one
   Proof: *We show: if* $\phi(aG_s) = \phi(bG_s)$, *then* $aG_s = bG_s$

$$\begin{aligned} \phi(aG_s) &= \phi(bG_s) \\ as &= bs \\ s &= \left(a^{-1}b\right) s, \quad \text{so } a^{-1}b \text{ stabilizes } s \\ a^{-1}b &\in G_s \text{ and by the rules for cosets} \\ aG_s &= bG_s. \end{aligned}$$

   (b) onto
      i. **Proof:** *We show, if* $s' \in O_s$ *then there is a coset* gG_s *for which* $\phi(gG_s) = s'$

$$\begin{aligned} s' &\in O_s \text{ so there is a } g \in G \text{ with} \\ s' &= gs, \quad \text{so } gG_s \text{ is a coset of } G_s \text{ and} \\ s' &= gs = \phi(gG_s), \\ &\text{Thus, } \phi \text{ is an onto map.} \end{aligned}$$

3. (15 points) Use a group action to count the rotational symmetries of a cube. Be explicit about what you choose as your set S.

Proof: *Let S be the six faces of the cube and $s \in S$ one face. Then the stabilizer of s, O_s, consists of the four rotations by multiples of $\pi/4$ about an axis the center and perpendicular to s. The orbit of s is all six faces of the cube since it is clear that any face can be rotated to any other face by a symmetry of the cube. Thus, if G is the group of rotational symmetries of the cube, then*

$$\begin{aligned} |G| &= |G_s| \cdot |O_s| \\ &= 4 \cdot 6 \\ &= 24. \end{aligned}$$

4. (10 points) Do **one** of the following.

(a) Prove if $|G| = p$ where p is a prime number, then G is isomorphic to a cyclic group of order p.

Proof: *Let G be a group of order p where p is a prime and let a be any non-identity element of G. Then, by Lagrange's Theorem, $|\langle a \rangle|$ divides $|G| = p$. But since $a \in \langle a \rangle$, $|\langle a \rangle| > 1$ and the primality of p yields $|\langle a \rangle| = p$. Since the only subgroup of G that has the same order as G is G itself, we conclude $\langle a \rangle = G$.*

(b) Determine all automorphisms of the group C_4. Be sure to show your functions are automorphisms.

Proof: *Let $\phi : C_4 \to C_4$ be an automorphism. Here $C_4 = \left\{\rho_0, \rho_{\pi/2}, \rho_\pi, \rho_{3\pi/2}\right\}$ is the cyclic group of rotations about the origin of order 4. Clearly the orders of the various elements of C_4 are: $|\rho_0| = 1$, $\left|\rho_{\pi/2}\right| = 4$, $|\rho_\pi| = 2$, $\left|\rho_{3\pi/2}\right| = 4$. Thus the automorphism ϕ must take ρ_0 and ρ_π to themselves. However, as far as orders of elements are concerned, ϕ might take the remaining two elements either to themselves or interchange them. We see that either possibility is possible since the two mappings: ϕ_1 which takes each element a to itself and ϕ_2 which takes every element to its inverse are both automorphisms and are, in fact, the only automorphisms of C_4.*

5. (15 points) Do **one** of the following.

(a) Let G be a subgroup of M that contains rotations by $\theta = \pi$ about two points: the origin and the point with coordinates $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Prove **algebraically** that G contains a translation. [See the Useful Facts at the end of the examination for tools.]

Proof: *Let $a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then, rotations in G about the origin have the form ρ_θ and rotations about the point a have the form $t_a \rho_\theta t_a^{-1}$. Since G contains rotations of size π about both points, then, by closure, G contains the element*

$$\begin{aligned} (\rho_{-\pi}) \left(t_a \rho_\pi t_a^{-1}\right) &= \rho_{-\pi} \left(t_a \rho_\pi\right) t_a^{-1} \\ &= \rho_{-\pi} \left(\rho_\pi t_{a'}\right) t_a^{-1} \\ &= \left(\rho_{-\pi} \rho_\pi\right) t_{a'} t_a^{-1} \\ &= t_{a'} t_{-a}. \end{aligned}$$

In this last form, the element is clearly a translation.

(b) Show **algebraically** that the successive reflection across two different lines through the origin is a rotation. For your proof, use the specific lines that form angles of $\pi/4$ and $\pi/2$ with the positive x_1- axis. What is the angle θ for the resulting rotation ρ_θ? [See the Useful Facts at the end of the examination for tools.]

Proof: *Reflection about the lines with angles* $\pi/4$, $\pi/2$ *are given by* $\rho_{\pi/2}r$ *and* $\rho_\pi r$, *respectively. Then*

$$\begin{aligned}\left(\rho_{\pi/2}r\right)(\rho_\pi r) &= \rho_{\pi/2}(r\rho_\pi)r\\ &= \left(\rho_{\pi/2}\rho_{-\pi}\right)(rr)\\ &= \rho_{-\pi/2}.\end{aligned}$$

Thus, the composition of the two reflections (in the order given) is equivalent to rotating around the origin clockwise by an angle of $\pi/2$.

6. (10 points) Find all matrices in the stabilizer of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ if the group action is conjugation in $GL(2, R)$. A useful fact is that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Proof: *We wish to find all matrices* A *for which* $A\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. *That is, find all matrices* A *for which*

$$\begin{aligned}A\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}A\\ \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\\ \begin{bmatrix} a & 2b \\ c & 2d \end{bmatrix} &= \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix}.\end{aligned}$$

From here it is easy to see that a *and* d *can be any real numbers but* $b = c = 0$. *Thus, the stabizer of* $s = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ *is the set* $O_s = \left\{\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right\}$ *of diagonal* 2×2 *matrices.*

1.2 Useful Facts

- **Theorem 1** *Every rigid motion can be written in one of the forms (uniquely)* $m = t_a\rho_\theta$ *or* $m = t_a\rho_\theta r$ *by using the following formulas for composition.*

  1. $t_a t_b = t_{a+b}$
  2. $\rho_\theta\rho_\eta = \rho_{\theta+\eta}$
  3. $rr = i$
  4. $\rho_\theta t_a = t_{a'}\rho_\theta$, *where* $a' = \rho_\theta(a)$
  5. $rt_a = t'_a r$, *where* $a' = r(a)$
  6. $r\rho_\theta = \rho_{-\theta}r$.