Mathematics 433 **Fall 2000**

Second set of Additional Problems

1. An n th root of unity is a complex number z such that $z^n = 1$. Prove that the n th roots of unity form a cyclic subgroup of order n of the group $G = (C, \times)$.

2. Do the following.

   (a) Prove that in any group, the orders of ab and ba are the same.

   (b) Describe all groups G that contain no proper subgroups.

   (c) Let G be a cyclic group of order n and let r be an integer dividing n. Prove that G contains exaclty one subgroup of order r.

3. Prove that the additive group of real numbers is isomorphic to the multiplicative group of postive reals.

4. Prove that the products ab and ba are conjugate elements in a group.

5. Let a, b be elements of a group G, and let $a' = bab^{-1}$. Prove that $a = a'$ if and only if a and b commute.

6. Do:

   (a) Let $b' = aba^{-1}$. Prove that $(b')^n = ab^n a^{-1}$.

   (b) Prove that if $aba^{-1} = b^2$, then $a^3ba^{-3} = b^8$.

7. Prove that the matrices $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ are conjugate elements in the group $GL(2, R)$ but they are not conjugate when regarded as elements of $SL(2, R) = \{A \in GL(2, R) : \det(A) = 1\}$.

8. Prove that the map $\phi : GL(n, R) \to GL(n, R)$ defined by $\phi(A) = (A^t)^{-1}$ is an automorphism.

9. Let G be a group with law of composition written $x\#y$. Let H be a group with law of composition $u \circ v$. What is the condition for a map $\phi : G \to H$ to be a homomorphism?

10. Let $\phi : G \to G'$ be a group homomorphism. Prove that for any elements $a_1, \cdots, a_k$ of G, $\phi(a_1 \cdots a_k) = \phi(a_1) \cdots \phi(a_k)$.

11. Describe all homomorphisms $\phi : (Z, +) \to (Z, +)$. Determine which are one-to-one, which are onto and which are isomorphisms.

12. Find all subgroups of S_3 and determine which of these are normal.

13. Find all subgroups of the quaternion group and determine which of these are normal.

14. Prove that the composition $\phi \circ \psi$ of homomorphisms is again a homomorphism. describe the kernel of $\phi \circ \psi$.

15. Do:

    (a) Let H be a subgroup of G and let $g \in G$. The **conjugate subgroup** gHg^{-1} of G is defined to be teh set of all conjugates ghg^{-1} where $h \in H$. Prove that $gHg^{-1}is$ a subgroup of G.

    (b) Prove that a subgroup H of G is normal in G if and only if $gHg^{-1} = H$ for all $g \in G$.

16. Let N be a normal subgroup of G and let $g \in G$, $n \in N$. Prove that $g^{-1}ng \in N$.

17. Let ϕ, ψ be two homomorphisms from a group G to another group G' and let $H \subset G$ be teh subset $\{x \in G : \phi(x) = \psi(x)\}$. Prove or disprove: H is a subgroup of G.

18. Prove that the center of a group is a normal subgroup.

19. Prove that the center of $GL(n, R)$ is teh subgroup $Z = \{cI_n : c \in R, c \neq 0\}$.

20. Prove that if a group contains exactly one element of order 2, then that element is in the center of the group.

21. Prove by giving an explicit example that $GL(2, R)$ is not a normal subgroup of $GL(2, C)$.

22. Let $\phi : G \to G'$ be an onto homomorphism and let N be a normal subgroup of G. Prove that $\phi(N)$ is a normal subgroup of G'.