Mathematics 433 **Fall 2006**

Extra Problem Set 01

Matrices

1. Find a formula for $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^n$, and prove it by induction.
2. Compute the products of a number of different pairs of matrices by block multiplication.
3. A square matrix A is called **nilpotent** if there is some positive integer k where $A^k = O$. Prove if A is nilpotent, then $I + A$ is invertible.
4. Do
   (a) Find infinitely many matrices B such that $BA = I_2$ where $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 2 & 5 \end{bmatrix}$.
   (b) Prove there is no matrix C with $AC = I_3$. (what happens if the matrix A had been **square?**)
5. The **trace** of a square matrix is the sum of its diagonal entries $\operatorname{trace}(A) = a_{11} + \cdots + a_{nn}$.
   (a) Show that $\operatorname{trace}(A+B) = \operatorname{trace}(A) + \operatorname{trace}(B)$ and $\operatorname{trace}(AB) = \operatorname{trace}(BA)$
   (b) Show that if B is invertible, then $\operatorname{trace}(A) = \operatorname{trace}\left(BAB^{-1}\right)$.
6. (*) Show that the reduced row echelon form obtained by row reduction on a matrix A is uniquely determined by A.
7. Prove that if the product AB of $n \times n$ matrices is invertible, then so are the factors A and B. Is this still true if A and B are not square?
8. Prove the Theorem: If A **is square** and has either a left or right inverse, then it also has the other.
9. Evaluate a number of determinants by hand using
   (a) Laplace expansion by minors,
   (b) Elementary matrices
10. Use induction to compute the following determinants

    (a) $\begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdots & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix}$

    (b) $\begin{bmatrix} 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & 2 & -1 & & \\ & & & \ddots & & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & \end{bmatrix}$

11. Consider the permutation p defined by $p(1) = 3$, $p(2) = 1$, $p(3) = 4$, $p(4) = 2$.

    (a) Find the associated permutation matrix P.

    (b) Write p as a product of transpositions (permutations that interchange exactly two elements) and evaluate the corresponding matrix product.

    (c) Compute the sign of p.

12. Prove every permutation matrix is the product of transpositions. [A transposition on a set S is a permutation that swaps exactly two elements of S. A transposition matrix, is a permutation matrix associated with a transposition.]

13. Prove that every matrix with a single 1 in each row and a single 1 in each column is a permutation matrix.

14. Let p be a permutation. Prove that $\text{sign}(p) = \text{sign}(p^{-1})$.

15. Prove that the transpose of a permutation matrix P is its inverse.

16. What is the permutation matrix associated with the permutation $p(i) = n - i, \quad 1 \le i \le n$?

17. Compute the adjoints of a number of matrices and verify the Theorem: $(adj(A))A = det(a)I$. [This problem is self-checking.]

18. (Vandermonde Determinant)

    (a) Prove that $\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = (b-a)(c-a)(c-b)$.

    (b) (∗) Prove an analogous formula for $n \times n$ matrices by using induction and row operations (in a clever fashion) to clear out the first column.

19. Consider a system of n linear equations in n unknowns: $AX = B$, where A and B have **integer** entries. Prove or disprove the following.

    (a) The system has a rational solution if $\det(A) \neq 0$.

    (b) If the system has a rational solution, then it also has an integer solution.

20. (∗) Let A, B be $m \times n$ and $n \times m$ matrices. Prove $I_m - AB$ is invertible if and only if $I_n - BA$ is invertible. [Hint: Use null spaces.]

21. An n th root of unity is a complex number z such that $z^n = 1$. Prove that the n th roots of unity form a cyclic subgroup of order n of the group $G = (C, \times)$.

22. Do the following.

    (a) Prove that in any group, the orders of ab and ba are the same.

    (b) Describe all groups G that contain no proper subgroups.

    (c) Let G be a cyclic group of order n and let r be an integer dividing n. Prove that G contains exaclty one subgroup of order r.