Mathematics 433-A Fall 2007

Exam 2

November 20, 2007

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Name

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Directions: Only write on one side of each page.

Do any five (5) of the following

1. Prove the First Isomorphism Theorem: Let $\phi : G \to G'$ be a surjective group homomorphism, and let $N = \ker(\phi)$. Then G/N is isomorphic to the G' by the function $\psi : G/N \to G'$ defined by $\psi(aN) = \phi(a)$. Recall that $\pi : G \to G/N$ defined by $\pi(g) = gN$ is a homomorphism (called the canonical homomorphism of N in G).

2. Let N be subroups of a group G where N is a normal subgroup of G.

   (a) Determine the subgroups of the restrictions of the canonical homomorphism $\pi : G \to G/N$ to the subgroups H and HN.

   (b) Apply the First Isomorphism Theorem to these restrictions to prove the Second Isomorphism Theorem: $H/(H \cap N)$ is isomorphic to $(HN)/N$.

3. When we classified the group M of rigid motions of the plane we claimed the following six relations were all true and proved a few of them. Add to our certainty by **algebraically** one of part d, part e, or part f.

   (a) $t_a t_b = t_{a+b}$

   (b) $\rho_\theta \rho_\eta = \rho_{\theta+\eta}$

   (c) $rr = i$

   (d) $\rho_\theta t_a = t_{a'} \rho_\theta$, where $a' = \rho_\theta(a)$

   (e) $r t_a = t'_a r$, where $a' = r(a)$

   (f) $r \rho_\theta = \rho_{-\theta} r$.

4. The following patterns represent small portions of two tilings of the infinite plane. Circle one of the following patterns and let G be the group of symmetries of that tiling. Determine, with brief explanation, both the point group of G and pictures of the vector(s) that form a basis for the translation group of G.

5. Let S be a set on which a group G operates. Let $H = \{g \in G : gs = s \text{ for all } s \in S\}$. Prove H is a subgroup of G and that it is a normal subgroup.

6. Let $G = GL(2, \mathbf{R})$ act on the set $S = \mathbf{R}^2$ by left multiplication. That is, if $A \in GL(2, \mathbf{R})$ and $x \in \mathbf{R}^2$, the group action is defined by $(A, x) \mapsto Ax$.

   (a) Prove that this really is a group action.

   (b) What is the stabilizer of $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

Figure 1: 1 2

7. Compute the glide vector of the glide $t_{\vec{a}}\rho_\theta r$ in terms of $\vec{a}$ and θ.

8. Given a group G, a subgroup H and any element $g \in G$, prove that the conjugate subgroup gHg^{-1} is isomorphic to H

9. Use a group action to count the rotational symmetries of a tetrahedron. Be explicit about what you choose as your set S.

Useful Facts

- $$\begin{aligned} \rho_\theta\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ r\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ t_{\vec{a}}(\vec{x}) &= \vec{x} + \vec{a} \end{aligned}$$