Smith Math 290 Spring 2010

Final Exam **May** 10, 2010

I affirm this work abides by the university's Academic Honesty Policy.

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Print Name, then Sign

Directions:

- **Only write on one side of each page.**
- **Use terminology correctly.**
- **Partial credit is awarded for correct approaches so justify your steps.**

Do two (2) of these "Computational" problems

C.1. [15 points] Using anything you know about determinants, compute the determinant of the following matrix **by hand.**

$$\begin{bmatrix} 0 & 0 & 1 & -1 & -1 \\ 2 & 4 & 2 & 4 & 2 \\ 2 & 4 & 3 & 0 & 3 \\ 3 & 6 & 6 & 3 & 6 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

C.2. [9, 6 points] Recall that the zero vector of the vector space $F(\mathbf{C}, \mathbf{C}) = \{f \mid f : \mathbf{C} \longrightarrow \mathbf{C}\}$ is the function $Z : \mathbf{C} \longrightarrow \mathbf{C}$ defined by $Z(x) = 0$ for all $x \in \mathbf{C}$. Consider the span $V = \left\langle \left\{ e^{kx} \mid k \in \mathbf{C} \right\} \right\rangle$ which is a subspace of $F(\mathbf{C}, \mathbf{C})$.

1. (a) Prove that $W = \{f \in V \mid f'' - 3f' + f = Z\}$ is a subspace of V.
   (b) Find a basis for W.

C.3. [15 Points] Find a basis for the kernel of the linear transformation $T : M_{2,2} \to M_{2,2}$ defined by $T(A) = \frac{1}{2}A - \frac{1}{2}A^t$.

Do any two (2) of these "Similar to In Class, Text, or Homework" problems

M.1. [15 Points] Prove that if $T : U \to V$ is a linear transformation and W is a subspace of U then the image of W under T, $T(W) = \{T(\vec{u}) : \vec{u} \in W\}$, is a subspace of V.

M.2. [15 Points] Prove Theorem SMEE (Similar Matrices have Equal Eigenvalues)

1. Suppose A and B are similar matrices. Then the characteristic polynomials of A and B are equal, that is, $\rho_A(x) = \rho_B(x)$.

M.3. [15 Points] Prove Theorem EER Eigenvalues, Eigenvectors, Representations: Suppose that $T : V \to V$ is a linear transformation and $B = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ is a basis of V. Then $\mathbf{v} \in V$ is an eigenvector of T for the eigenvalue λ if and only if $\rho_B(\mathbf{v})$ is an eigenvector of $M^T_{B,B}$ for the eigenvalue λ.

Do two (2) of these "Other" problems

T.1. [15 Points] In Proof LT-1 you saw a one-step test for whether or not a function is a linear transformation. This problem gives a one-step test for showing a subset of a vector space is a subspace.

1. Prove that a subset W of a vector space V is a suspace if and only if $\alpha\vec{w}_1 + \beta\vec{w}_2 \in W$ is true for all $\vec{w}_1, \vec{w}_2 \in W$ and for all $\alpha, \beta \in \mathbf{C}$.

T.2. [15 Points] Let $B = \left\{e^x, xe^x, x^2e^x\right\}$ be a basis for the subspace V of the vector space of functions with domain and codomain the set of complex numbers: $F(\mathbf{C}, \mathbf{C}) = \{f \mid f : \mathbf{C} \to \mathbf{C}\}$

1. (a) Find the matrix representation $M^T_{B,B}$ of the linear transformation $T : V \to V$ defined by $T(f) = f'$.
   (b) Use this matrix representation to find the kernel of T, $\ker(T)$.

T.3. [15 Points] It is a true fact that if $V = \{A \in M_{n,n} \mid A \text{ is symmetric}\}$ and $W = \{B \in M_{n,n} \mid B \text{ is skew-symmetric}\}$ then $M_{n,n} = V \oplus W$. Prove this fact in the special case when $n = 2$.

T.4. [15 points] Professor Beezer has proven that if V is a finite-dimensional vector space and $T : V \to V$ has $Range(T) = V$ then T is an isomorphism. Show that this is not necessarily the case if V is infinite dimensional by giving an example of a linear transformation $T : P \to P$ that is not injective but that has $Range(T) = P$.Be sure to explain why your example has the desired properties. [Recall that P is the infinite dimensional vector space of **all** polynomials.]

You must do both of these problems ON THIS SHEET

R.1. [15 points] Prove that the set $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{C}^3 : \; 5x_1 - 7x_2 - 2x_3 = 0 \right\}$ is a subspace of $\mathbf{C}^3$ by applying the three-part test of Theorem TSS. Write your proof according to the standards of this semester's writing exercises.

R.2. [15 points] Suppose that $Z : V \longrightarrow V$ is the linear transformation denoted by $Z(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$ (i.e. Z is the "zero" linear transformation). Suppose that $T : V \longrightarrow V$ is a linear transformation such that $T^4 = Z$ (where $T^4 = T \circ T \circ T \circ T$). Then prove that T is not invertible. Write your proof according to the standards of this semester's writing exercises.