Mathematics 280-A Spring 2008

Final Exam

May 14, 2008

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Technology used: ______________________________

Only write on one side of each page.

- Show all of your work. Calculators may be used for numerical calculations and answer checking only.

Problems

1. Do **one** (1) of the following.
   (a) Write an equation for either of the planes that are parallel to the plane $x + 2y - 2z = 6$ and are six units away from it.
   (b) Do the lines $x = -4t - 1,\ y = 2t + 4,\ z = 12t + 2$ and $x = -2t + 9,\ y = 3t - 9,\ z = t - 8$ intersect? If so, find the point(s) of intersection. Be careful, t may mean something different in each parametrization.

2. Prove the following differentiation formula. That is, if f and g are differentiable functions (scalar fields) of x and y, then
$$\vec{\boldsymbol{\nabla}}\left(\frac{f}{g}\right) = \frac{g\vec{\boldsymbol{\nabla}} f - f\vec{\boldsymbol{\nabla}} g}{g^2}.$$

3. Do any **two** (2) of the following.
   (a) Show that the following limit does not exist.
$$\lim_{(x,y)\to(0,0)} \frac{\left(x^2 + y\right)^2}{x^4 + y^2}.$$
   (b) A closed box is found to have length 2 feet, width 4 feet, and height 3 feet where the measurement of each dimension is made with a maximum possible error of ± 0.02 feet. The top of the box is made from material that costs $\$2/\text{ft}^2$ and the material for the sides and bottom costs $\$1.50/\text{ft}^2$. Use differentials to bound the possible error involved in the computation of the cost of the box?
   (c) Assume that $h(x, y)$, $f(x)$, and $g(x)$ are differentiable. Use chain rule methods to show any function of the form
$$z = h(x, t) = f(x + at) + g(x - at)$$
   is a solution of the *wave equation*
$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}.$$

4. Find and classify the local maximum, local minimum, and saddle points of

$$f(x,y) = \frac{x^3}{3} - 4xy + 9y^3.$$

5. The base of an open top aquarium with given volume V_0 is made of slate and the sides are made of glass. If slate costs five times as much (per unit area) as glass, find the dimensions of the aquarium that minimize the cost of the materials. Why are you sure this is really a minimum?

6. Find the volume of the solid that the cylinder $r = 4\cos\theta$ cuts out of the sphere of radius 4 centered at the origin.

7. Rewrite (but do not evaluate) the triple integral

$$\int_{-1}^{0}\int_{0}^{y^2}\int_{0}^{1} dx\,dz\,dy$$

in the orders

   (a) $dz\ dy\,dx$
   (b) $dy\,dz\,dx$

8. Do **one** (1) of the following.

   (a) Find the work done by the force field $\mathbf{F}(x,y) = 2x^2\,\mathbf{i} + xy\,\mathbf{j}$ on a particle that moves once around the circle $x^2+y^2=4$ oriented in the counterclockwise direction.
   (b) Is $\mathbf{F}(x,y,z) = (2xz + \sin(y))\,\mathbf{i} + x\cos(y)\ \mathbf{j} + (x^2+5)\ \mathbf{k}$ a conservative vector field? If so, find a potential function $f(x,y,z)$. Show your work.
   (c) Let D be simply connected region bounded by the smooth curve C oriented counterclockwise.
      i. Use Green's Theorem to explain why the area A of region D satisfies

$$A = \frac{1}{2}\oint_C x\ dy - y\ dx$$

9. Do **one** (1) of the following where $\vec{\mathbf{F}}$ is a vector field on a connected and simply connected domain whose component functions have continuous first partial derivatives.

   (a) Prove: If the curl of $\vec{\mathbf{F}}$ is $\vec{\mathbf{0}}$ then $\vec{\mathbf{F}}$ is conservative. That is, if $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{F}} = \vec{\mathbf{0}}$ then $\vec{\mathbf{F}}$ is conservative.
   (b) Use triple integrals in cylindrical or spherical coordinates to write out [**but do not evaluate**] iterated integrals that give the 'hyper-volume' $\iiiint_D 1\,dH$ of a hypersphere $D = \{x,y,z,w : x^2+y^2+z^2+w^2 = R^2\}$. Here R is a constant and and the result of projecting ("smashing") the hypersphere into xyz-space is the three dimensional sphere $x^2+y^2+z^2=R^2$.
   (c) Make an appropriate change of variables and evaluate

$$\iint_R [(x-y)\cos(x+y)]\ \ dA$$

   where D is the region bounded by the lines $x-y=1$, $x-y=2$, $x+y=0$ and $x+y=\frac{\pi}{2}$.