Mathematics 280 **Spring 2008**

Exam 4

April 17, 2008

KEY

Technology used: ______________________

Only write on one side of each page.

- Show all of your work. Calculators may be used for numerical calculations and answer checking only.

Problems

1. LIke number 31 in section 12.5. [15 points] Let f be a function on two variables that has continuous partial derivatives of all orders and consider the points $A(1,3), B(3,4), C(1,6)$ and $D(6,14)$. The directional derivative of f at A in the direction of the vector $\overrightarrow{AB}$ is $3\sqrt{2}$, and the directional derivative at A in the direction of $\overrightarrow{AC}$ is 25. Find the directional derivative at A in the direction of the vector $\overrightarrow{AD}$.

   Solution: $\overrightarrow{AB} = \langle 2,1\rangle$, $\overrightarrow{AC} = \langle 0,3\rangle$, $\overrightarrow{AD} = \langle 5,11\rangle$ and recall directional derivatives use require unit vectors.

   (a) $D_{\vec{u}_1} f(A) = \langle f_x(A), f_y(A)\rangle \cdot \frac{1}{\sqrt{5}}\langle 2,1\rangle = \frac{2}{\sqrt{5}} f_x + \frac{1}{\sqrt{5}} f_y = 3\sqrt{2}$

   (b) $D_{\vec{u}_2} f(A) = \langle f_x(A), f_y(A)\rangle \cdot \frac{1}{3}\langle 0,3\rangle = f_y = 25$

   (c) $f_y = 25$ implies $f_x = \left(3\sqrt{2} - \frac{25}{\sqrt{5}}\right)\frac{\sqrt{5}}{2} = \frac{3\sqrt{10}-25}{2}$

   (d) So $D_{\vec{u}_3} f(A) = \langle f_x(A), f_y(A)\rangle \cdot \frac{1}{146}\langle 5,11\rangle = \frac{3\sqrt{10}-25}{2}\frac{5}{\sqrt{146}} + 25\frac{11}{\sqrt{146}} \approx 19.5494648425137$

2. [15 points] Do **one** (1) of the following.

   (a) This is number 18 in section 12.6. Find parametric equations for the line tangent to the curve of intersection of the surfaces $x^2+y^2=4$ and $x^2+y^2-z=0$ at the point $\left(\sqrt{2},\sqrt{2},4\right)$.

   Solution: The direction vector of the line we seek is perpendicular to the normal vectors of the tangent planes to both surfaces at the point $\left(\sqrt{2},\sqrt{2},4\right)$.

   i. The normal vector to $x^2+y^2-4=0$ at the given point is $\left\langle 2\sqrt{2},2\sqrt{2},0\right\rangle$ and the normal vector to the second is $\left\langle 2\sqrt{2},2\sqrt{2},-1\right\rangle$

   ii. The cross product of these two normal vectors is $\left\langle 2\sqrt{2},2\sqrt{2},0\right\rangle \times \left\langle 2\sqrt{2},2\sqrt{2},-1\right\rangle = \left\langle -2\sqrt{2},2\sqrt{2},0\right\rangle = -2\sqrt{2}\langle -1,1,0\rangle$ so we use $\vec{d} = \langle -1,1,0\rangle$ for the direction vector of the line.

   iii. The parametrized line is $\vec{r}(t) = \left\langle \sqrt{2}-t, \sqrt{2}+t, 4+0t\right\rangle$

(b) This is like number 96 on page 783. Around the point $(1,0)$ in the plane

   i. Is $f(x,y) = x^2(y+1)$ more sensitive to changes in x or to changes in y? Why?
   Solution: Since $f_x(1,0) = 2$ and $f_y(1,0) = 1$ then changes in x have about twice the effect on the outputs of f than do similar changes in y.

   ii. What ratio of dx to dy will make df equal zero at $(1,0)$.
   Solution: At the point $(1,0)$ we have $df = \vec{\nabla} f(1,0) \cdot \langle dx, dy \rangle = 2dx + dy$ and this equals zero precisely when the ratio of dx to dy is $-1:2$.

3. This is like number 51 in section 12.7. [15 points] Find the absolute maximum and minimum values, if they exist, of

$$f(x,y) = 2x^3 + y^4$$

where the domain is the set $D = \{(x,y) : y^2 \leq 1 - x^2\}$.

Solution: We are guaranteed both an absolute minimum and an absolute maximum since f is continuous on this closed bounded domain.

(a) On the interior of the domain we check that $\vec{\nabla} f(x,y) = \langle 6x^2, 4y^3 \rangle = \langle 0,0 \rangle$ only when $(x,y) = (0,0)$. This is the only critical point.

(b) On the boundary of the domain we have $y^2 = 1 - x^2$, $-1 \leq x \leq 1$ which gives us

$$\begin{aligned} f\left(x, \pm\sqrt{1-x^2}\right) &= 2x^3 + \left(1 - x^2\right)^2, \quad -1 \leq x \leq 1 \\ &= 2x^3 + 1 - 2x^2 + x^4 \\ f' &= 4x^3 + 6x^2 - 4x \\ &= 2x(x+2)(2x-1) \end{aligned}$$

(c) Thus we need to consider $x = 0, -2, \frac{1}{2}$ because they make f'equal zero as well as $x = -1, 1$ as endpoints. The corresponding y values are given by $y = \pm\sqrt{1-x^2}$.

   i. -2 is not in the domain $-1 \leq x \leq 1$

   ii. when $x = 0$, $y = \pm 1$ yielding $(0,1)$ and $(0,-1)$

   iii. when $x = \frac{1}{2}$, $y = \pm\frac{\sqrt{3}}{2}$ yielding $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

   iv. when $x = \pm 1$, $y = 0$ yielding $(1,0)$ and $(-1,0)$.

(d) Checking our list the maximum value of f is $f(1,0) = 2$ and the minimum is $f(-1,0) = -2$.

4. [15 points] Do **one** (1) of the following using the method of Lagrange multipliers.

(a) This is like number 11 in section 12.8. Find the dimensions of the rectangle of largest area that can be inscribed in the ellipse $x^2/9 + y^2/25 = 1$ with sides parallel to the coordinate axes. What is the largest area?

Solution: We maximize the continuous function $f(x,y) = 4xy$ on the closed bounded domain $\{(x,y) : g(x,y) = x^2/9 + y^2/25 - 1 = 0\}$ and then compute $|f(x,y)| = |4xy|$ to find the maximum area.

   i. Our system of equations is: $4y = \frac{2}{9}x\lambda$, $4x = \frac{2}{25}y\lambda$, $x^2/9 + y^2/25 = 1$

A. If either x or y is zero so is the other (by the first two equations) but $(0,0)$ is not in the domain so we can assume $(x,y)\neq(0,0)$.

B. For $x,y\neq 0$

$$\begin{aligned} 4y &= \frac{2}{9}x\lambda \\ &= \frac{2}{9}\frac{2}{4\cdot 25}y\lambda^2 \\ (30)^2 y &= \lambda^2 y \\ &\text{and} \\ 4x &= \frac{2}{25}y\lambda \\ 4x &= \frac{2}{25}\frac{2}{9}x\lambda^2 \\ (30)^2 x &= \lambda^2 x \end{aligned}$$

C. This tells us that $\lambda=\pm 30$ and for these values of λ we have the system of equations

$$\begin{aligned} 4y &= \frac{2}{9}(\pm 30)\,x \quad \text{and} \\ 4x &= \frac{2}{25}(\pm 30)\,y \end{aligned}$$

both equations of which simplify to

$$y=\pm\frac{5}{3}x$$

D. Plugging this into our constraint equation we get $\frac{x^2}{9}+\frac{1}{25}\frac{25}{9}x^2=1$ which tells us that $2x^2=9$ so $x=\frac{3}{\sqrt{2}}$.Since $y=\frac{5}{3}x$, this yields the four point $\left(\pm\frac{3}{\sqrt{2}},\pm\frac{5}{\sqrt{2}}\right)$

ii. Checking our list of points we see that $f\left(\pm\frac{3}{\sqrt{2}},\pm\frac{5}{\sqrt{2}}\right)=4\left(\pm\frac{3}{\sqrt{2}}\right)\left(\pm\frac{5}{\sqrt{2}}\right)=\pm 30$

iii. Thus the absolute maximum area is $|\pm 30|$ and occurs at all four points $\left(\pm\frac{3}{\sqrt{2}},\pm\frac{5}{\sqrt{2}}\right)$.

(b) This is number 22 in section 12.8. Find the point(s) on the surface whose equation is $xyz=1$ closest to the origin. Although this set is unbounded, you may use the geometric fact that there **is** an absolute minimum value.

Solution: We minimize $f(x,y,z)=x^2+y^2+z^2$ on the closed (but unbounded) domain $\{(x,y,z):\ g(x,y,z)=xyz-1=0\}$.

i. Our system of four equations is $2x=yz\lambda$, $2y=xz\lambda$, $2z=xy\lambda$ and $xyz=1$.

ii. Note first that if one of x,y, or z is 0, then the first three equations tell us that so are the other two. Since $(0,0,0)$ does not satisfy the last equation we know that none of x,y and z can be zero and we can safely divide by them.

iii. The first three equations tell us that $2x^2=2y^2=2z^2=xyz\lambda=(1)\,\lambda$. From this we see the last equation becomes that $\pm x=\pm y=\pm z$ so that, for these points, $g(\pm x,\pm y,\pm z)=\pm x^3=1$ so that each of x,y,z must equal either $+1$ or -1. This gives us 8 points to check $(\pm 1,\pm 1,\pm 1)$.

iv. Plugging these into the function f we get $f(\pm 1, \pm 1, \pm 1) = 1 + 1 + 1 = 3$. Since we are told there is an absolute minimum value of f, it must be 3 and it occurs at all 8 of our points. Note that there is no absolute maximum of this function.

5. This is out of section 13.3 and as such should not have been on the exam. I graded it as an optional problem out of six total. [15 points] Compute the average value of $f(x, y) = x \sin(xy)$ over the rectangle $R = [0, \pi/2] \times [0, 1]$.

   Solution: The average value is

$$\begin{aligned} & \frac{1}{\text{Area}(D)} \iint_D f(x, y)\ dA \\ = & \frac{1}{(\pi/2 - 0)(1 - 0)} \int_0^{\pi/2} \int_0^1 x \sin(xy)\ dy\ dx \end{aligned}$$

   (a) Using $u = xy$ and $du = x\,dy$ for the inner integral we get $\int_0^1 x \sin(xy)\ dy = [-\cos(xy)]_{y=0}^1 = -\cos(x) + 1$

   (b) Thus, the average value is

$$\begin{aligned} \frac{1}{\pi/2} \int_0^{\pi/2} (1 - \cos(x))\ dx &= \frac{2}{\pi} [x - \sin(x)]_0^{\pi/2} \\ &= \frac{2}{\pi} \left[\left(\frac{\pi}{2} - 1 \right) - (0 - 0) \right] \\ &= 1 - \frac{2}{\pi}. \end{aligned}$$

6. This is number 30 in section 13.2. [15 points] Evaluate the double integral.

$$\int_0^3 \int_{\sqrt{x/3}}^1 e^{\left(y^3\right)}\ dy\ dx$$

   Solution: Since this order of integration is very difficult we reverse the order. By drawing a picture we can see the region $\left\{ (x, y) :\ 0 \le x \le 3,\ \sqrt{x/3} \le y \le 1 \right\}$ can also be described as $\left\{ (x, y) : 0 \le y \le 1,\ 0 \le x \le 3y^2 \right\}$.Thus by Fubini's theorem the given integral is equal to

$$\begin{aligned} \int_0^1 \int_0^{3y^2} e^{\left(y^3\right)}\ dx\ dy &= \int_0^1 \left[x e^{\left(y^3\right)} \right]_{x=0}^{3y^2}\ dy \\ &= \int_0^1 3y^2 e^{\left(y^3\right)} dy \\ &= \int_{u=0^3}^{1^3} e^u du \\ &= e - 1 \end{aligned}$$

7. This is like problems 35 − 44 in section 13.2. [10 points] Set up iterated integral(s) for the volume of the solid that remains when a square hole of side length 2 is drilled through the center of a sphere of radius $\sqrt{2}$.

   Solution: Looking down from the positive z-axis, as shown in the picture that was on the blackboardand using the symmetry of spheres, we see the volume we seek comes in four symmetric pieces. We find the volume of the top piece and multiply by four. Note that the points where the square intersects the circle in the xy-plane are $\left(\pm\sqrt{2}, \pm\sqrt{2}\right)$.

Since, the z values go from the bottom hemi-sphere of $x^2+y^2+z^2 = 2$ to the top hemi-sphere, our solid (one fourth of the total) is described as the set

$\left\{(x, y, z) : -\sqrt{2} \leq x \leq \sqrt{2}, 1 \leq y \leq \sqrt{2-x^2}, -\sqrt{2-x^2-y^2} \leq z \leq \sqrt{2-x^2-y^2}\right\}$. Thus the total volume remaining after the square hole is drilled out of the sphere is given by the triple integral

$$4\int_{-\sqrt{2}}^{\sqrt{2}}\int_{1}^{\sqrt{2-x^2}} \left[\sqrt{2-x^2-y^2} - \left(-\sqrt{2-x^2-y^2}\right)\right]\ dy\ dx$$