Mathematics 232-A **Final Exam** **Spring 2006**

May 10, 2006

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Name

Technology used: ____

Directions: Only write on one side of each page. Use terminology correctly. Partial credit is awarded for correct approaches so justify your steps.

Exam 5

"Computational" Problems

C.1. Do **one** (1) of the following:

(a) Find a basis for the kernel of the linear transformation $T : P_2 \to C^1$ given by

$$T(f) = \int_0^1 f(x)\ dx.$$

(b) Find a basis for the range of the linear transformation $T : P_2 \to R^3$ given by

$$T(f) = \begin{bmatrix} f(0) \\ f'(1) \\ f(2) \end{bmatrix}.$$

C.2. Do **one** (1) of the following.

(a) Let $S : P_2 \longrightarrow P_3$ be given by $S(p) = x^3 p'' - x^2 p' + 3p$. Find the matrix representation of S with respect to the bases B, C where the basis for P_2 is $B = \{x+1, x+2, x^2\}$ and the basis for P_3 is $C = \{1, x, x^2, x^3\}$.

(b) Find the matrix $M^T_{B,B}$ of the linear transformation $T : P_2 \to P_2$ given by

$$T(f) = f(0) + f'(1)\, x^2$$

with respect to the ordered basis $\mathcal{B} = \{1, 1+x, 1+x+x^2\}$

Do Two (2) of these "In text, class or homework" problems

M.1. Prove Theorem MRCLT (Matrix Representation of a Composition of Linear Transformations): Suppose that $T : U \to V$ and $S : V \to W$ are linear transformations, B is a basis of U, C is a basis of V , and D is a basis of W. Then $M^{S\circ T}_{B,D} = M^{S}_{C,D} M^{T}_{B,C}$.

M.2. Prove Theorem KNSI (Kernel and Null Space Isomorphism): Suppose that $T : U \to V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V. Then the kernel of T is isomorphic to the null space of $M^T_{B,C}$, $K(T) \cong N(M^T_{B,C})$.

M.3. Prove Theorem ICBM (Inverse of Change-of-Basis Matrix): Suppose that V is a vector space, and B and C are bases of V. Then the change-of-basis matrix $C_{B,C}$ is nonsingular and $(C_{B,C})^{-1} = C_{C,B}$.

M.4. Prove Theorem EER (Eigenvalues, Eigenvectors, Representations): Suppose that $T : V \to V$ is a linear transformation and B is a basis of V . Then $v \in V$ is an eigenvector of T for the eigenvalue λ if and only if $\rho_B(v)$ is an eigenvector of $M^T_{B,B}$ for the eigenvalue λ.

Do One (1) of these "Not in Text" problems

T.1. Suppose $f : V \to W$ and $g : W \to V$ are functions (they do not need to be linear transformations) and that the composition of g with f is the identity function on V, $\quad g \circ f = I_V$.

(a) Prove f must be injective.

(b) Prove g must be surjective.

T.2. Consider the function $T : \mathbf{C}^2 \to \mathbf{C}^2$ that takes each vector v to its complex conjugate $\overline{v}$, $T(v) = \overline{v}$.

(a) Use the standard basis $B = \{e_1, e_2\}$ of $\mathbf{C}^2$ in both the domain and codomain of T to find the matrix representation of T, $M^T_{B,B}$.

(b) Use $v = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$ and your $M^T_{B,B}$ from above to show that the Fundamental Theorem of Matrix Representation (FTMR – stated below) fails.

(c) Explain this apparent contradiction.

Theorem FTMR, Fundamental Theorem of Matrix Representation: Suppose that $T : U \to V$ is a linear transformation, B is a basis for U, C is a basis for V and $M^T_{B,C}$ is the matrix representation of T relative to B and C. Then, for any $\vec{u} \in U$, $\rho_C(T(\vec{u})) = M^T_{B,C}(\rho_B(\vec{u}))$.

Cumulative Exam

Do Two (2) of these "In text, class or homework" problems

CC.1. Suppose $T : V \to V$ is a linear transformation. Prove that T is invertible only if the number 0 is not an eigenvalue of T.

CC.2. Let U, V be abstract vector spaces and $T : U \to V$ a function. Show that T is a linear transformation **if and only if** for all $\vec{u}_1, \vec{u}_2 \in U$ and all scalars a, b we have $T\left(a\vec{u}_1 + b\vec{u}_2\right) = aT\left(\vec{u}_1\right) + bT\left(\vec{u}_2\right)$

CC.3. Prove that in any linearly dependent set of vectors $S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ there is an index k such that the vector $\vec{v}_k$ can be written as a linear combination of the **previous** vectors in S.

CC.4. If A is diagonalizable, is A^T similar to A?

Do Two (2) of these "Not in text" problems

MM.1. Do **two** (2) of

(a) (1 point each) If A is a square matrix, make a list of statements we know are equivalent to "A is nonsingular". [Be careful, T is not a matrix and make sure you are proving the correct half of the "if and only if".]

(b) (10 points) An **orthogonal** matrix is an $n \times n$ matrix A satisfying $A^T A = I_n$. Prove the determinant of any orthogonal matrix must be either 1 or -1.

(c) (10 points)

i. If the rank of a 5×3 matrix A is 3, what is rref(A)?

ii. Consider a linear system of equations $A\overrightarrow{x} = \overrightarrow{b}$, where A is a 4×3 matrix. We are told that the rank of $\left[A \mid \overrightarrow{b}\right]$ is 4. How many solutions does this system have?

MM.2. Use the principle of mathematical induction to prove the following fact we have used repeatedly throughout the semester.

Suppose V is a vector space, α is a scalar and $\vec{v}_1, \vec{v}_2, \vec{v}_3, \cdots, \vec{v}_n$ are vectors in V. Then $\alpha\left(\vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_n\right) = \alpha\vec{v}_1 + \alpha\vec{v}_2 + \cdots + \alpha\vec{v}_n$ for every positive integer n.

MM.3. Given an invertible matrix S, prove the following transformation $T : M_{nn} \to M_{nn}$ is linear.

$$T\left(A\right) = S^{-1}AS$$

MM.4. If there are square matrices A and B satisfying the property that $B^2 = A$, then we say B is a **square root** of A. It is easy to see that a diagonal matrix $D = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & d_{nn} \end{bmatrix}$ has $\begin{bmatrix} \sqrt{d_{11}} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sqrt{d_{nn}} \end{bmatrix}$ as a square root.

Prove that if A is a diagonalizable matrix, then A has a square root.

You MUST do both of these problems.

Show your work on this page.

1. (10 points) Let V and W be vector spaces and let $T : V \to W$ be a linear transformation. Prove that $K\left(T\right)$, the kernel of T, is a subspace of V. Recall that $K\left(T\right) = \left\{\vec{v} \in V : T\left(\vec{v}\right) = \vec{0}\right\}$. Your solution to this problem will be graded in part based on the quality of the exposition. Write carefully and clearly, with complete explanations of your proof.

2. (10 points) Use contradiction to prove the following: If V is a vector space, W is a subspace of V and $\vec{v}_0 \in V$ but $\vec{v}_0 \notin W$, then for any $\vec{w} \in W$, $\vec{v}_0 + \vec{w} \notin W$. Your solution to this problem will be graded in part based on the quality of the exposition. Write carefully and clearly, with complete explanations of your proof.