Mathematics 232 Fall 1998 Exam 2

October 5, 1998

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Name

Technology used: ______________________

Textbook/Notes used: ______________________

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use a text or notes or technology. **Only write on one side of each page.**

The Problems

1. Assuming you already know that the reflection, in R^3, through the plane $y = z$ is a linear transformation. Determine the matrix A of this transformation.

2. Determine if the matrix A below is invertible and, if it is, find its inverse by hand (no technology).
$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

3. Assuming you already know that the reflection, in R^4 (thought of as $x, y, z, w-$space), through the hyperplane $y = z$ is a linear transformation. Determine the matrix A of this transformation.

4. Give an example of a matrix A such that image(A) is spanned by the vector $\begin{bmatrix} 2 \\ 7 \end{bmatrix}$.

5. If L is the line in the plane that contains the vector $\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$, find the matrix of the linear transformation that projects a vector $\overrightarrow{x}$ onto the line L.

6. If A is an invertible $n \times n$ matrix and B is an $n \times n$ matrix, Is it always the case that
$$\left(ABA^{-1}\right)^4 = AB^4A^{-1}?$$

7. Let $\overrightarrow{v}_1, \overrightarrow{v_2}, \overrightarrow{v_3}$ be three vectors in R^n. Show that span$(\overrightarrow{v}_1, \overrightarrow{v_2}, \overrightarrow{v_3})$ is a subspace of R^n.

8. Find the matrix A of the linear transformation $T : R^2 \to R^2$ with
$$T\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \text{ and } T\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

9. Suppose a line L in R^3 contains the unit vector $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$. Find the matrix of the linear transformation $T(\overrightarrow{x}) = \text{proj}_L \overrightarrow{x}$ by giving the entries of the matrix A in terms of the components of $\overrightarrow{u}$. What is the sum of the diagonal entries in A?

10. Let $T : R^n \to R^m$ be a function and c a scalar. Define a new function $(cT) : R^n \to R^m$ by

$$(cT)(\overrightarrow{x}) = cT(\overrightarrow{x}) \quad \text{for all } \overrightarrow{x} \text{ in } R^n.$$

Prove that if T is a linear transformation, then so is (cT).

11. Prove that a function $T : R^n \to R^m$ is a linear transformation if and only if for all scalars c, d and for all vectors $\overrightarrow{v}$, $\overrightarrow{w} \in R^n$ we have $T(c\overrightarrow{v} + d\overrightarrow{w}) = cT(\overrightarrow{v}) + dT(\overrightarrow{w})$.

12. Show that it is impossible for a matrix A to have two different inverses. [Hint: Pretend that B, C are distinct inverses and consider the product BAC.]

13. We showed in class that it is impossible to have matrices $A_{3\times2}$ and $B_{2\times3}$ where the product AB equaled I_3. On the other hand, it **is** possible to have such matrices where $BA = I_2$. Give an example of specific matrices A, B satisfying this last equality.

14. Suppose $A_{n\times m}$ and $B_{m\times n}$ are matrices such that $BA = I_m$. Let $\overrightarrow{b}$ be a particular vector in R^m. Show that the system of equations $B\overrightarrow{x} = \overrightarrow{b}$ must be consistent.

15. Is it possible to have an invertible 2×2 matrix A with the property that $A^2 = O_2$? Why or why not? (Here O_2 denotes the 2×2 zero matrix.)

16. Let $\overrightarrow{v}_1, \overrightarrow{v}_2, \cdots, \overrightarrow{v}_n$ be some vectors in R^n such that the matrix

$$S = \begin{bmatrix} | & | & & | \\ \overrightarrow{v}_1 & \overrightarrow{v}_2 & \cdots & \overrightarrow{v}_n \\ | & | & & | \end{bmatrix}.$$

is invertible. Let $\overrightarrow{w}_1, \overrightarrow{w}_2, \cdots, \overrightarrow{w}_n$ be any vectors in R^m. Use the fact (which you need not prove) that there is a unique linear transformation $T : R^n \to R^m$ such that $T(\overrightarrow{v}_i) = \overrightarrow{w}_i$, for all $i = 1, 2, \cdots, n$. Find the matrix A of this transformation in terms of S and

$$B = \begin{bmatrix} | & | & & | \\ \overrightarrow{w}_1 & \overrightarrow{w}_2 & \cdots & \overrightarrow{w}_n \\ | & | & & | \end{bmatrix}.$$

17. Prove, with the setup as in the problem above, that the linear transformation T really is **unique.**

18. Consider two matrices A and B whose product AB is defined. Describe the ith row of the product AB in terms of the rows of A and the matrix B.