Mathematics 221-B

December 17, 1998

Technology used: ____

Textbook/Notes used: ____

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use a text or notes or technology. **Only write on one side of each page.**

Select your problems to illustrate the state of your knowledge of multivariate calculus. If around half of your problems are "straightforward", you should probably have a total of $7-10$ problems.

Section 1: Questions on Chapter 13.

Do at least three 3 of these problems.

1. Find the work done by the force field $\mathbf{F}(x,y) = x^2\,\mathbf{i} + xy\,\mathbf{j}$ on a particle that moves once around the circle $x^2+y^2=4$ oriented in the **clockwise** direction.

2. The formulas for the center of mass $(\overline{x},\overline{y},\overline{z})$ of a thin wire with density function $\rho(x,y,z)$ in the shape of a curve C are:

$$m=\int_C \rho(x,y,z)\;ds,\ \overline{x}=\frac{1}{m}\int_C x\rho(x,y,z)\;ds,\ \overline{y}=\frac{1}{m}\int_C y\rho(x,y,z)\;ds,\ \overline{z}=\frac{1}{m}\int_C z\rho(x,y,z)\;ds.$$

   Find the x-coordinate of the center of mass of a wire in the shape of the helix $x=t,\ y=\cos(t),\ z=\sin(t),\ 0\le t\le 2\pi$ if the density at any point is equal to the square of the distance from the origin.

3. The force exerted on an electric charge at the origin on a charged particle at a point (x,y,z) with position vector $\overrightarrow{r} = < x,y,z >$ is

$$\mathbf{F}(\overrightarrow{r}) = \frac{K\overrightarrow{r}}{|\overrightarrow{r}|^3}$$

   where K is a constant. Find the work done as the particle moves along a straight line from $(2,0,0)$ to $(2,1,5)$.

4. Show that the vector field $\mathbf{F}(x,y) = < 2xy+\sin y, x^2+x\cos y >$ is conservative and use the fundamental theorem of line integrals to compute $\int_C \mathbf{F}\cdot\mathbf{d}\overrightarrow{r}$ where C is any curve that starts at $(-3,-3)$ and ends at $(1,1)$.

5. Green's Theorem:

Section 2

1. Does the line $x = 1 - t$, $y = 3t$, $z = 1 + t$ have any points in common with the sphere $2x^2 + 2y^2 + 2z^2 + x + y + z = 9$? If so, find all such points.

2. Find an equation of the plane that contains the line $\frac{x-5}{-1} = \frac{y+2}{4} = \frac{z-3}{2}$ and passes through the point $(1, 2, 3)$.

3. Find the unit normal vector $\mathbf{N}$ to the curve $3x^2 + 8xy + 2y^2 - 3 = 0$ at the point $(1, 0)$.

4. Describe the level curves of the function $f(x, y)$ given below and draw a reasonable sketch of its graph.
$$f(x, y) = \begin{cases} x^2 + y^2 - 1, & \text{if } x^2 + y^2 \geq 1 \\ 0, & \text{if } x^2 + y^2 \leq 1 \end{cases}$$

5. Show that every plane that is tangent to the cone $x^2 + y^2 = z^2$ passes through the origin.

6. Find the volume of the solid under the graph of $z = 3x^2 + y^2$ and above the region bounded by $y = x$ and $x = y^2 - y$.

7. Assuming that all functions are differentiable, show that any function of the form
$$z = f(x + at) + g(x - at)$$
is a solution of the *wave equation*
$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

8. The voltage V in a simple electrical circuit is slowly decreasing as the battery wears out. The resistance R is slowly increasing as the resistor heats up. Use Ohm's Law, $V = IR$, to find how the current I, is changing at the moment when $R = 400\ \Omega$, $I = 0.08\ A$, $dV/dt = -0.01\ \mathrm{V}/s$, and $dR/dt = 0.03\ \Omega/s$.

9. The temperature T of a metal ball, whose center is located at the origin, is inversely proportional to the distance from the center of the ball. The temperature at the point $(1, 2, 2)$ is $120°$. Find the rate of change of T at the point $(1, 2, 2)$ in the direction toward the point $(2, 1, 3)$.

10. Show that the product of the x-, y-, and z-intercepts of any tangent plane to the surface $xyz = c$ is a constant. [Here c is a constant.]

11. Find the volume of the solid inside both the cylinder $x^2 + y^2 = 4$, and the ellipsoid $4x^2 + 4y^2 + z^2 = 64$.

12. Rewrite the iterated triple integral below in the given orders.

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\; dy\, dz\, dx$$

   (a) $\iiint f(x, y, z)\; dx\, dz\, dy$

   (b) $\iiint f(x, y, z)\; dz\, dy\, dx$

13. Sketch the quadric surface given by $z = x^2 + xy + y^2 - 4x - 4y + 2$. [Hint: Use what you know about critical points.]

14. Use Clairaut's Theorem to show that if f and all first, second and third partial derivatives are continuous, then $f_{xxy} = f_{xyx}$.

15. At what point on the curve $x = t^3$, $y = 3t$, $z = t^4$ is the normal plane parallel to $6x + 6y - 8z = 1$?

16. Find the volume of the largest rectangular box with edges parallel to the axes that can be inscribed in the sphere $x^2 + y^2 + z^2 = 9$.

17. The base of an aquarium with given volume V is made of slate and the sides are made of glass. If slate costs five times as much (per unit area) as glass, find the dimensions of the aquarium that minimize the cost of the materials.

18. A particle of mass m in a rectangular box with dimensions x, y, z has ground state energy

$$E(x, y, z) = \frac{k^2}{8m}\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right)$$

   where k is a physical constant. If the volume of the box is fixed (say $V_0 = xyz$), find the values of x, y, z that minimize the ground state energy.

19. Use triple integrals to find the 'hyper-volume' of a hypersphere $x^2 + y^2 + z^2 + w^2 = R^2$. Here R is a constant and and the result of "smashing" the hypersphere into xyz-space is the three dimensional sphere $x^2 + y^2 + z^2 = R^2$.

20. Use double integrals to find a formula for the surface area of the frustum of the cone $z = 4\sqrt{x^2 + y^2}$ between the planes $z = 4$ and $z = 1$.

21. Show that arc length is independent of the parametrization of the curve by letting $\overrightarrow{r}_1(t)$, $a \le t \le b$ and $\overrightarrow{r}_2(u)$, $\alpha \le u \le \beta$ be two parameterization of a curve C where $t = g(u)$ and $g'(u) > 0$ for all values of u. Then show that $L_1 = L_2$ when $L_1 = \int_a^b |\overrightarrow{r}'(t)|\, dt$ and $L_1 = \int_\alpha^\beta |\overrightarrow{r}'(u)|\, du$.

22. Find the average value of ρ over the solid ball $\rho \le 3$. Is this reasonable geometrically? Explain.

23. Sketch the solid that is the domain of integration. **Do not evaluate.**

$$\int_{\pi/6}^{5\pi/6} \int_{\csc\phi}^{2\csc\phi} \int_{0}^{2\pi} \rho \sin^2\phi \, d\theta \, d\rho \, d\phi$$

24. Draw the region R in the xy-plane that corresponds to the square S in the uv-plane with corners (0,0), (0,1),(1,0),(1,1) for the change of variables $x = u\cos(v)$, $y = u\sin(v)$ [There are only three corners] and use the Jacobian to find the area of region R.

25. Make the change of variables $u = x + 2y$, $v = x - y$ and evaluate

$$\iint_R \frac{x+2y}{\cos(x-y)} \, dA$$

where R is the parallelogram bounded by the lines $y = x$, $y = x - 1$, $x + 2y = 0$, $x + 2y = 2$.