

October 5, 2004

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 Name

**Technology used:** \_\_\_\_\_ **Directions:**

Be sure to show all steps in your solutions. Partial credit is based on your work – not on your answer. Include a careful sketch of any graph obtained by technology in solving a problem. **Only write on one side of each page.**

### The Problems

LAST YEAR

1. (4, 4, 7 points)

(a) Evaluate  $\int \frac{1}{x+5} dx$

(b) Evaluate  $\int (x + 1/2)e^{x^2+x} dx$

(c) What is the average value of the volumes of all possible spheres with radii between 3 and 6? [Useful fact: The volume of a sphere of radius  $r$  is  $4/3\pi r^3$ .]

2.

3. (10, 5 points)

(a) Find the general solution to the following first-order, nonlinear, separable differential equation with initial condition.

$$\frac{dy}{dx} = \frac{x^2 \sqrt{y^2 + 5}}{y \sqrt{x^3 + 1}}$$

(b) Find the particular solution corresponding to the initial condition  $y(2) = 2$ .4. (10, 10 points) The mass density of oil, measured in kilograms per square meter, in a circular oil slick on the surface of the ocean at a distance  $r$  meters from the center of the slick is given by  $\rho(r) = \frac{50}{1+r} \frac{\text{kg}}{\text{m}^2}$ .(a) Suppose the slick extends from  $r = 0$  to  $r = 10,000$  m. Using a partition with  $n$  subintervals and either the left or right endpoints, write a Riemann sum approximating the total mass of oil in the slick. [Mass is measured in kilograms (kg).](b) Write the definite integral that is equal to the limit as  $\|P\| \rightarrow 0$  of your Riemann sum. Do not evaluate this definite integral.5. Do **one** of the following.(a) (5, 5, 10 points) Suppose  $f(x)$  is a monotone increasing function on the interval  $[a, b]$ . Briefly explain why each of the following is true.i. For any integer  $n$ , the right endpoint approximation  $R_n$  is an overestimate of the value of  $\int_a^b f(x) dx$ .

- ii. For any integer  $n$ , the left endpoint approximation  $L_n$  is an underestimate of the value of  $\int_a^b f(x) dx$ .
- iii. If we use the average,  $\frac{R_n+L_n}{2}$ , of  $R_n$  and  $L_n$  as an estimate for  $\int_a^b f(x) dx$ , then the error in our approximation can be no worse than  $\frac{R_n-L_n}{2}$ . That is,

$$\left| \int_a^b f(x) dx - \frac{R_n + L_n}{2} \right| \leq \frac{R_n - L_n}{2}.$$

6. (20 points) Do one of the following

- (a) Find the derivative  $H'(x)$  of

$$H(x) = x^2 \int_5^{\cos(x)} \frac{\ln(t)}{t^4 + 7} dt.$$

- (b) Find a function  $f$  that satisfies the equation

$$\tan(x) + e^x = \int_5^x \sqrt{f(t) - 2} dt.$$

Hint: take the derivative of both sides.

## The Problems

1. Match the slope fields on the accompanying sheet with the differential equations

- (a)  $y'(t) = t^2 - y^2$  : Slope field \_\_\_\_\_
- (b)  $y'(t) = t^2 - y$  : Slope field \_\_\_\_\_
- (c)  $y'(t) = t - y^2$  : Slope field \_\_\_\_\_

2. ( 15 points ) 2 of 4

- (a) Find the orthogonal trajectories of the family of functions given by  $y = x^4 + C$ .
- (b) (10 points) Use separation of variables to find the general solution of

$$g'(x) = \frac{x^2}{\cos(g(x))}.$$

Then find the specific solution that satisfies the initial condition  $g(2) = \frac{\pi}{2}$ .

- (c) Show that the function  $y = \frac{1}{x} \int_1^x \frac{e^t}{t} dt$  is a solution of the differential equation  $x^2 y' + xy = e^x$ .
- (d) At a certain moment a 100-gallon mixing tank is full of brine containing 0.25 pounds of salt per gallon. If the brine is being continuously drawn off at the rate of 4 gallons per minute and replaced by brine containing 0.2 pounds of salt per gallon, explain why the following differential equation accurately models the flow of fluid through the tank. Where  $S(t)$  represents the number of pounds of salt in the tank at time  $t$ .

$$S'(t) = (.2)(3) - 4 \frac{S(t)}{100}$$

The solution of the differential equation above is

$$S(t) = 20 + Ce^{-.04t}, \quad S(0) = 25.$$

How much salt is present 25 minutes later.

3. Molten glass is flowing through a rectangular opening of base width 5 cm at a rate of 3 cm/s. At time  $t$  s the depth of the glass is  $H(t)$  cm. Using the ideas associated with definite integrals as the limit of Riemann sums, explain why the volume of glass passing through the opening in the time period from  $t = a$  to  $t = b$  is given by:  $\int_a^b 15H(t) dt$ . Keep track of units in your analysis.
4. Do **one** of the following.

- (a) Find a function  $f$  that satisfies the equation

$$\cos(x) + x^2 = \int_2^x \sqrt{3t^2 + tf(t)} dt.$$

[Hint: take the derivative of both sides of the equation.]

- (b) If  $F(x) = \int_1^x f(t) dt$  where  $f(t) = \int_1^{t^2} \frac{\sqrt{1+u^2}}{u} du$ , find  $F''(2)$ .

5. Evaluate 5 of the following indefinite and definite integrals.

- (a)

$$\int \frac{1}{x-5} dx$$

- (b)

$$\int \frac{23}{\sqrt{1-x^2}} - \sec(x) \tan(x) dx$$

- (c)

$$\int \frac{dx}{x \ln(x)}$$

- (d)

$$\int \frac{\arctan(2x)}{1+4x^2} dx$$

- (e)

$$\int \frac{(3x+6) dx}{(x^2+2x-3)^{\frac{2}{3}}}$$

- (f)

$$\int_0^{0.5} \frac{dx}{\sqrt{1-x^2}}$$

- (g)

$$\int \frac{5-2x}{\sqrt{1-x^2}} dx$$

6. The following formula is from a table of integrals.

$$\int \sqrt{a^2 - u^2} du = \frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \arcsin\left(\frac{u}{a}\right) + C$$

(a) Use this formula to evaluate  $\int_0^5 \sqrt{25 - 4x^2} dx$ .

7. The following sum is the 1000'th term of a sequence  $A_n$  that is a 'discrete antiderivative' of the sequence  $a_n = 3n + 1$ . Evaluate the following sum. Use our technique for finding 'discrete derivatives' and discrete antiderivatives' to find a formula for the general term  $A_n$ .

$$\sum_{i=1}^{1000} 3i + 1$$

8. On another planet, the acceleration due to gravity is only 10 m/s<sup>2</sup>. If a stone is dropped from a structure 1000 meters above ground level, how long (in seconds) will it take the stone to reach the ground?

9. Given the function  $f(x) = e^{x^2}$ .

(a) Show the work justifying  $f''(x) = (2 + 4x^2)e^{x^2}$ .

(b) What is the smallest value of the integer  $n$  necessary to guarantee that the Trapezoid rule will approximate the integral

$$\int_0^1 e^{x^2} dx$$

to within  $\epsilon = 10^{-16}$ ?

10. ( 15 points ) Most of us know the number  $e$  is approximately 2.72. Use the fact that  $e = 1 + \int_0^1 e^x dx$  and the error bound for Simpson's Rule,  $E_n$ , to determine the number of subintervals necessary to determine the value of  $e$  accurate to within  $10^{-9}$ .

$$E_n \leq \frac{1}{180} \frac{(b-a)^5}{n^4} K$$

11. Determine the values of  $n$  for which a theoretical error of less than 0.01 can be guaranteed if the following integral is estimated using the Trapezoid Rule.

$$\int_2^4 \ln(x) dx$$

12. Explain why Simpson's Rule with  $n = 2$  would give an exact answer (no error) for the integral  $\int_0^{100} (2x^3 - 5x^2 + 23) dx$ .

13. On May 7, 1992, the space shuttle *Endeavor* was launched on mission STS-49. The table below, provided by NASA, gives the velocity data for the shuttle between liftoff and the jettisoning of the solid rocket boosters. Use the data in the table to form a Riemann sum that estimates the height above the earth's surface of the space shuttle *Endeavor*, 62 seconds after liftoff.

Event	Time (s)	Velocity (ft/s)
Launch	0	0
Begin roll maneuver	10	185
End roll maneuver	15	319
Throttle to 89%	20	447
Throttle to 67%	32	742
Throttle to 104%	59	1325
Maximum dynamic pressure	62	1445
Solid rocket booster separation	125	4151