Introduction to Graph Theory

Math 434

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Preliminary Definitions

Definition: A **graph** $G = (V, E)$ is a mathematical structure consisting of two sets V and E The elements of V are called *vertices*, and the elements of E are called *edges*. Each edge has a set of one or two vertices associated to it, which are called its *endpoints*.

Definition: A **self-loop** is an edge that joins a single endpoint to itself.

Definition: A **proper edge** is an edge that is not a self-loop.

Definition: A **multi-edge** is a collection of two or more edges having identical endpoints.

Definition: **Adjacent vertices** are two vertices that are joined by an edge.

Definition: **Adjacent edges** are two edges that have an endpoint in common.

Definition: If vertex v is an endpoint of edge e, then v is said to be **incident** on e, and e is incident on v.

Definition: The **degree** of a vertex v in a graph G, denoted $\deg(v)$ is the number of proper edges incident on v plus twice the number of self-loops.

Definition: A **directed edge** is an edge, one of whose endpoints is designated as the *tail*, and whose other endpoint is designated as the *head*.

Definition: A **directed graph** (digraph) is a graph each of whose edges is directed.

Definition: A **weighted graph** is a graph in which each edge is assigned a number, called its *edge-weight*.

Definition: A graph or digraph is **simple** if it has neither self-loops nor multi-edges.

Definition: The **edge-complement** of a simple graph G, denoted G^c, is the graph on the same vertex-set, such that two vertices are adjacent in G^c if and only

Definition: A **complete graph** is a simple graph such that every pair of vertices is joined by an edge. Any complete graph on n vertices is denoted K_n.

Definition: A **regular graph** is a graph whose vertices all have equal degree. A **k-regular** graph is a regular graph whose common degree is k.

Definition: A **path** from vertex v_0 to vertex v_n is an alternating sequence of vertices and edges

$$P =< v_0, e_1, v_1, e_2, ..., v_{n-1}, e_n, v_n >$$

such that the endpoints of e_i are v_{i-1}, v_i and there are no repeated edges or vertices (except possibly the initial and final vertices).

Definition: The **length** of a path is the number of edges in the path.

Definition: A graph is **connected** if for every pair of vertices u and v, there is a path from u to v.

Definition: A path starting and ending at the same vertex is a **cycle** (except for self loops).

Definition: An **acyclic** graph has no cycles.

Graph Representations

Definition: The **incidence matrix** of a graph G is the matrix I_G whose rows and columns are indexed by some orderings of V_G and E_G, respectively, such that

$$I_G[v, e] = \left\{ \begin{array}{cc} 0 & \text{,if } v \text{ is not an endpoint of } e \\ 1 & \text{, if } v \text{ is an endpoint of } e \\ 2 & \text{, if } e \text{ is a self-loop at } v \end{array} \right|$$

Definition: The **adjacency matrix** of a graph G, denoted A_G, is the matrix whose rows and columns are both indexed by identical orderings of V_G, such that

$$A_G[u, v] = \left\{ \begin{array}{c} \text{the number of edges between } u \text{ and } v \text{ if } u \neq v \\ \text{the number of self-loops at } v \text{ if } u = v \end{array} \right|$$

Graph Structure

Definition: A **subgraph** of a graph G is a graph H whose vertices and edges are all in G. If H is a subgraph of G, we may also say that G is a *supergraph* of H.

Definition: For a given graph G, the subgraph **induced** on a vertex subset U of V_G, denoted by $G(U)$, is the subgraph of G whose vertex-set is U and whose edge-set consists of all edges in G that have both endpoints in U. That is,

$$V_{G(U)} = U \qquad \text{and} \qquad E_{G(U)} = \{e \in E_G : \{\text{the endpoints of } e\} \subseteq U\}$$

Definition: If v is a vertex of a graph G, then the **vertex-deletion subgraph** $G - v$ is the subgraph induced by the vertex-set $V_G - \{v\}$. That is,

$$V_{G-v} = V_G - \{v\} \qquad \text{and} \qquad E_{G-v} = \{e \in E_G : v \notin \{\text{endpoints of } e\}\}$$

Definition: If e is an edge of a graph G, then the **edge-deletion subgraph** $G - e$ is the subgraph induced by the edge-set $E_G - \{e\}$. That is,

$$V_{G-e} = V_G \qquad \text{and} \qquad E_{G-e} = E_G - \{e\}$$

Definition: A **graph isomorphism** $f : G \to H$ is a pair of one-to-one, onto functions

$$f_V : V_G \to V_H \qquad \text{and} \qquad f_E : E_G \to E_H$$

such that for every edge $e \in E_G$, the function f_V maps the endpoints of e to the endpoints of the edge $f_E(e)$.
Note: Graph isomorphism preserve incidence and edge multiplicity.
Note: The relation $\approx$ ("isomorphic to") is an equivalence relation.
Definition: Each equivalence class under $\approx$ is called an **isomorphism type**.
Definition: A **numerical graph invariant** is a numerical property of all graphs, such that any two isomorphic graphs have the same value.
Some graph invariants:

1. The number of vertices
2. The number of edges
3. The length of a particular path
4. For any possible subgraph, the number of distinct copies
5. For a simple graph, the edge-complement

The Graph-Reconstruction Problem

Definition: Let G be a graph with $V_G = \{v_1, v_2, ..., v_n\}$. Then the **vertex-deletion subgraph list** of G is the list of the subgraphs $G - v_1, ..., G - v_n$.
Definition: The **graph-reconstruction problem** is to decide whether two non-isomorphic graphs with three or more vertices can have the same vertex-deletion subgraph list.
Note: Originally formulated by P.J. Kelly and S.M. Ulam in 1941, this problem is among the foremost unsolved problems in graph theory, because it is concerned with a list of *unlabeled* graphs.
The Reconstruction Conjecture: Let G and H be two graphs with at least three vertices and with $V_G = \{v_1, v_2, ..., v_n\}$ and $V_H = \{w_1, w_2, ..., w_n\}$, such that $G - v_i \approx H - w_i$, for each $i = 1, ..., n$. Then $G \approx H$.

Automorphisms

Review: A **graph isomorphism** $f : G \to H$ is a pair of one-to-one, onto functions

$$f_V : V_G \to V_H \qquad \text{and} \qquad f_E : E_G \to E_H$$

such that for every edge $e \in E_G$, the function f_V maps the endpoints of e to the endpoints of the edge $f_E(e)$.
Definition: An **automorphism** of a graph G is an isomorphism f that maps G to itself.
Terminology: The functions f_V and f_E of a graph automorphism are referred to as the **vertex-permutation** and the **edge-permutation**, respectively.
Note: The vertex-permutation of an automorphism on a simple graph completely determines the automorphism.
Theorem: Let f and g be automorphisms of the same simple graph G, such that $f_V = g_V$. Then $f_E = g_E$.
Review: Let P be a nonempty collection of permutations on the same finite set of objects Y such that P is closed under composition. Then the mathematical structure $[P : Y]$ is a **permutation group**.
Theorem: The set of all automorphisms of a graph is a permutation group.
Definition: The **automorphism group** of a graph G is the permutation group of all automorphisms of the graph G. It is denoted $Aut(G)$. Its restrictions to V_G and to E_G are denoted $Aut_V(G)$ and $Aut_E(G)$, respectively.

Cycle Index of a Permutation Group

Definition: The **cycle structure of a permutation** is the number of cycles of each length in its disjoint cycle form.
Definition: Let $\mathbf{P} = [P : Y]$ be a permutation group on a set Y of n objects, and let $\pi \in P$. The **cycle-structure monomial** of π is the n-variable monomial

$$\zeta(\pi) = \prod_{k=1}^{n} z_k^{r_k} = z_1^{r_1} z_2^{r_2} ... z_n^{r_n}$$

where z_k is a formal variable and r_k is the number of k-cycles in the disjoint cycle form of π.
Definition: Let $\mathbf{P} = [P : Y]$ be a permutation group on a set of n objects. Then the **cycle index polynomial** of $\mathbf{P}$ is the polynomial

$$Z_{\mathbf{P}}(z_1, ..., z_n) = \frac{1}{|P|} \sum_{\pi \in P} \zeta(\pi)$$

Review of Burnside's Lemma
Definition: Let $\mathbf{P} = [P : Y]$ be a permutation group, and let $y \in Y$. The **stabilizer** of y is the subgroup $Stab(y) = \{\pi \in P : \pi(y) = y\}$.
Definition: Let $\mathbf{P} = [P : Y]$ be a permutation group, and let $\pi \in P$. The **fixed-point set** of a permutation π is the subset $Fix(\pi) = \{y \in Y : \pi(y) = y\}$.
Definition: For a given automorphism π on a graph, the **fixed-vertex set** and **fixed-edge set** are the fixed-point sets of π_V and π_E, respectively.
Definition: Let $\mathbf{P} = [P : Y]$ be a permutation group. The **orbit** of an object $y \in Y$ is the set $\{\pi(y) : \pi \in P\}$ of all the objects onto which y is permuted.
Lemma: Let $\mathbf{P} = [P : Y]$ be a permutation group. Then

$$\sum_{y \in Y} |Stab(y)| = \sum_{\pi \in P} |Fix(\pi)|$$

Lemma: Let $\mathbf{P} = [P : Y]$ be a permutation group and $y \in Y$. Then

$$Stab(y) = \frac{|P|}{|orbit(y)|}$$

Lemma: Let $\mathbf{P} = [P : Y]$ be a permutation group with n orbits. Then

$$\sum_{y \in Y} \frac{1}{|orbit(y)|} = n$$

Lemma (Burnside's): Let $P = [P : Y]$ be a permutation group with n orbits. Then

$$n = \frac{1}{|P|} \sum_{\pi \in P} |Fix(\pi)|$$

Graph Colorings
Definition: A k**-coloring** of a set Y is a mapping f from Y onto the set $\{1, 2, ..., k\}$, in which the number $f(y)$ is called the **color** of y. Any k-coloring of Y is also called a **coloring**.
Definition: A **($\leq k$)-coloring** of Y is a coloring that uses k or fewer colors.
Definition: Let $\mathbf{P} = [P : Y]$ be a permutation group acting on a set Y, and let f and g be ($\leq k$)-colorings of Y. They are **P-equivalent colorings** if there is a permutation $\pi \in P$ such that $g = f\pi$, that is, if for every object $y \in Y$, the color $g(y)$ is the same as the color $f(\pi(y))$.
Definition: Let G be a graph. Two vertex-colorings are **equivalent vertex-colorings** if they are $Aut_V(G)$-equivalent.
Definition: Let G be a graph. Two edge-colorings are **equivalent edge-colorings** if they are $Aut_E(G)$-equivalent.
Notation: The set of all ($\leq k$)-colorings of the elements of a set Y is denoted $Col_k(Y)$. The set of $\mathbf{P}$-equivalence classes of $Col_k(Y)$ is denoted $\{Col_k(Y)\}_P$.
Counting Vertex- and Edge- Colorings
Proposition: Let $\mathbf{P} = [P : Y]$ be a permutation group acting on a set Y. Let $f \in Col_k(Y)$ be a ($\leq k$)-coloring of Y, and let $\pi \in P$ be a permutation in P. Then the composition $f\pi$ of permutation π followed by coloring f is a coloring in $Col_k(Y)$.
Corollary: Let $\mathbf{P} = [P : Y]$ be a permutation group acting on a set Y, and let $\pi \in P$. Then the mapping $\pi_{YC} : Col_k(Y) \to Col_k(Y)$ defined by

$\pi_{YC}(f) = f\pi$, for every coloring $f \in Col_k(Y)$ is a permutation on the set $Col_k(Y)$.
Definition: The mapping defined above is called the **induced permutation action** of π on $Col_k(Y)$.
Definition: Let $\mathbf{P} = [P : Y]$ be a permutation group acting on a set Y. The collection $\mathbf{P_C} = [P : Col_k(Y)]$ of induced permutations on $Col_k(Y)$ is called the **induced permutation group**.
Theorem: Let $\mathbf{P} = [P : Y]$ be a permutation group. Then

$$|[Col(Y)]_P| = Z_{\mathbf{P}}(k, ..., k)$$

Counting Simple Graphs
Step 1: Calculate the Cycle Index of $Aut_V(K_n)$
Step 2: Calculate the Cycle Index of $Aut_E(K_n)$
Step 3: Evaluate the Cycle Index of $Aut_E(G)$
Pőlya Substitution
Definition: Let $Z_{\mathbf{P}}(z_1, ..., z_n)$ be the cycle-index polynomial for a permutation group $\mathbf{P} = [P : Y]$ on a set Y of n objects. The **Pőlya substitute** of order k for the cycle-index variable z_j is the k-variate polynomial

$$x_1^j + x_2^j + ... + x_k^j$$

Definition: Let $Z_P(z_1, ..., z_n)$ be the cycle-index polynomial for a permutation group $\mathbf{P} = [P : Y]$ on a set Y of n objects. The **Pőlya-counting polynomial** of order k is the polynomial obtained by replacing each cylce-index variable z_j by its Pőlya substitute or order k. It is denoted $Z_{\mathbf{P}}(x_1 + ... + x_k)$.
Binary Operations on Graphs
Definition: The **(graph) union** of two graphs $G = (V, E)$ and $G' = (V', E')$ is the graph $G \cup G'$ whose vertex set and edge-set are the disjoint unions, respectively, of the vertex-sets and edge-sets of G and G'.
Definition: The **(cartesian) product** $G \times H$ of the graphs G and H has as its vertex-set the cartesian product

$$V_{G\times H} = V_G \times V_H$$

and as its edges a union of two products:

$$E_{G\times H} = (V_G \times E_H) \cup (E_G \times V_H)$$

The endpoints of the edge (u, d) are the vertices (u, x) and (u, y), where x and y are the endpoints of edge d in graph H. The endpoints of the edge (e, w) are (u, w) and (v, w), where u and v are the endpoints of edge e in graph G.
Definition: The **join** $G + H$ of the graphs G and H is obtained from the graph union $G \cup H$ by adding an edge between each vertex of G and each vertex of H.
Definition: Let G and H be disjoint graphs, with $u \in V_G$ and $v \in V_H$. The **vertex amalgamation** $(G \cup H)/\{u = v\}$ is the graph obtained from the union $G \cup H$ by merging vertex u of graph G and vertex v of graph H into a single vertex called u. The vertex-set of this new graph is $V_G \cup V_H - \{v\}$ and the edge-set is $E_G \cup E_H$, except that vertex u replaces vertex v as an endpoint of any edge of H in which v occurs.
Traversals and Trees
Definition: A **depth-first traversal** is when we visit the starting node and then proceed to follow links through the graph until we reach a dead end. When we reach a dead end, we back up along the path until we find an unvisited adjacent edge.
Definition: A **breadth-first traversal** is when we visit the starting node and then on the first pass visit all of the nodes directly connected to it. In the second pass, we visit the nodes that are two edges away from the starting node.
Definition: A subgraph H is said to **span** a graph G if $V_H = V_G$.
Definition: A **tree** is a connected graph that has no cycles.
Theorem: Let T be a tree where v_i and v_j are arbitrary vertices. Then v_i and v_j are connected by exactly one proper path. Also, in the graph formed by adding the edge $\{v_i, v_j\}$ to T, there is exactly one cycle.
Theorem: In an (n, m) tree, $m = n - 1$.
Definition: A **spanning tree** of a graph is a spanning subgraph that is a tree.
Proposition: A graph G is connected if and only if it has a spanning tree.
Definition: A **minimum spanning tree** (MST) of a weighted connected graph is a spanning subgraph of the original graph so that the subgraph is connected and the total of the edge weights is the smallest possible.

Definition: A graph is said to be **biconnected** if there is at least two distinct paths between any two vertices.

Definition: A **biconnected component** of a graph is the set of three or more vertices for which there are at least two paths between each vertex. A bicomponent can also have just two nodes and one edge connecting them.

Definition: An **articulation point** of a graph is a vertex that shared by two biconnected components.

Puzzles

Crossing the River:

A man (m) wishes to transfer his three pets - a dog (d), a cat (c), and a rabbit (r) - from the left bank of a river to the right bank. He wishes to do so by swimming back and forth between the banks, carrying with him one pet at a time. However, he cannot leave the dog alone with the cat, nor the cat alone with the rabbit. How can he accomplish the transfer under these conditions?

Divide the Wine:

We are given three jugs, a, b, and c, with capacities 8, 5, and 3 quarts, respectively. Jug a is filled with wine, and our objective is to divide its contents into two equal parts by pouring the wine from one jug to another (without resorting to any measuring devices other than the three jugs). Each move will consist of either completely filling or completely emptying one of the three jugs.

References

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