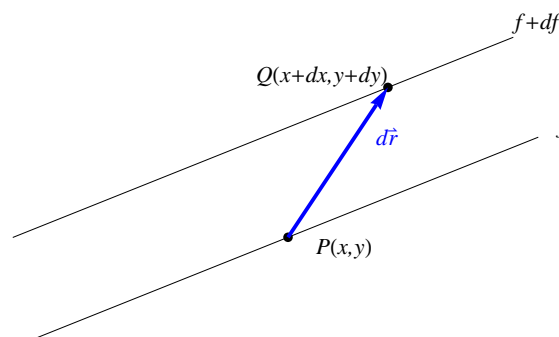


Components of the gradient vector

- start with function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and point P in the domain where, in a zoomed-in view, the level curve through P and nearby level curves are parallel lines
- define *gradient vector* $\vec{\nabla} f$ as vector that
 - points in direction of greatest rate of change (so perpendicular to level curve through P)
 - has magnitude $\|\vec{\nabla} f\|$ equal to that greatest rate of change
- introduce cartesian coordinates to have $P(x, y)$
- consider infinitesimal displacement $d\vec{r} = dx \hat{i} + dy \hat{j}$ consisting of displacements dx and dy in the x and y directions, respectively
- for the displacement $d\vec{r}$, there is a corresponding infinitesimal change df in the function values



- relate df to dx and dy using partial derivatives as

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

- note that each term on the right side is a contribution to the change df that has the form
(rate of change in f with respect to change in coordinate) \times (size of change in coordinate)
- factor this using the dot product as

$$df = \left(\frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} \right) \cdot (dx \hat{i} + dy \hat{j})$$

- in this product of two vectors, the first vector has information about rate of change and the second vector has information about displacement

- for convenience, name the first vector in the product $\vec{\text{Bob}}$ so have

$$\vec{\text{Bob}} = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j}$$

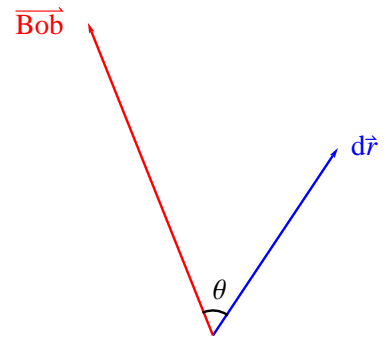
and can write

$$df = \vec{\text{Bob}} \cdot d\vec{r} \quad (1)$$

- now argue that $\vec{\text{Bob}}$ is equal to the gradient vector $\vec{\nabla} f$
- start by writing the geometric expression for the dot product in (1) to get

$$df = \|\vec{\text{Bob}}\| \|d\vec{r}\| \cos \theta = \|\vec{\text{Bob}}\| ds \cos \theta$$

where θ is the angle between $\vec{\text{Bob}}$ and $d\vec{r}$ and $ds = \|d\vec{r}\|$



- the rate of change in f for a displacement $d\vec{r}$ is the ratio of df to ds
- dividing through by ds in the previous relation gives

$$\text{rate of change in } f \text{ for displacement } d\vec{r}: \frac{df}{ds} = \|\vec{\text{Bob}}\| \cos \theta \quad (2)$$

- now consider all displacements $d\vec{r}$ having the same magnitude $ds = \|d\vec{r}\|$ while allowing the direction to vary so the only variable in (2) is θ
- since $\cos \theta$ has values between -1 and 1 , the greatest rate of change is for $\cos \theta = 1$ corresponding to $\theta = 0$
- so, the greatest rate of change is for a displacement in the direction of $\vec{\text{Bob}}$ with magnitude

$$\left. \frac{df}{ds} \right|_{\theta=0} = \|\vec{\text{Bob}}\| \cos 0 = \|\vec{\text{Bob}}\| (1) = \|\vec{\text{Bob}}\|$$

- in other words, $\vec{\text{Bob}}$ is a vector that
 - points in direction of greatest rate of change
 - has magnitude equal to that greatest rate of change

- thus, $\vec{\text{Bob}}$ is equal to the gradient vector $\vec{\nabla} f$
- recalling the definition of $\vec{\text{Bob}}$, we have

$$\vec{\nabla} f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j}$$

- this result gives us a way to compute the components of a gradient vector $\vec{\nabla} f$ if we have a formula for f in terms of cartesian coordinates
- knowing that $\vec{\text{Bob}} = \vec{\nabla} f$, can relate df to $\vec{\nabla} f$ by rewriting (1) as

$$df = \vec{\nabla} f \cdot d\vec{r}$$