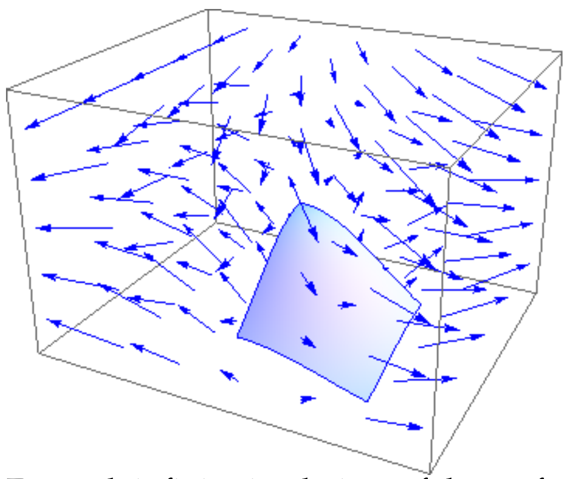
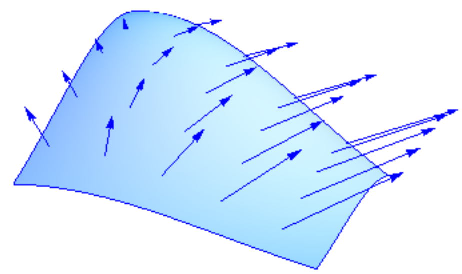
Integrating a vector field over a surface

Definition

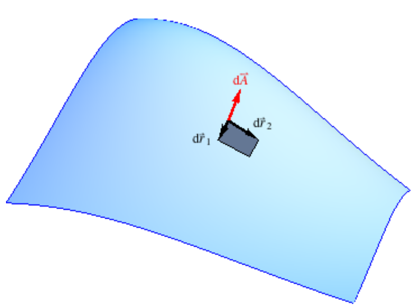
We are given a vector field $\vec{F}$ in space and an oriented surface S in the domain of $\vec{F}$ as shown in the figure below on the left. The figure on the right shows the vector field plotted only at points on the surface. The general idea of integrating the vector field $\vec{F}$ over the surface S is

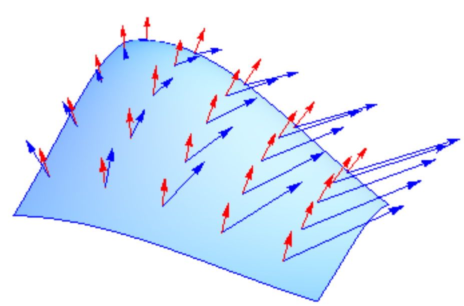
add up, over the surface, infinitesimal contributions of the form
(component of $\vec{F}$ normal to S) times (area of piece of S).





For each infinitesimal piece of the surface, we have an infinitesimal area vector $d\vec{A}$ as shown in the figure below on the left. For each infinitesimal piece of the surface, the corresponding infinitesimal area vector points perpendicular to the surface (at that piece) and its magnitude gives the area of that piece. A sampling of these along the surface is shown in the figure on the lower right along with the vector field along the surface.





From our knowledge of dot product, we know that $\vec{F} \cdot d\vec{A}$ gives us (component of $\vec{F}$ normal to S) times (area of piece of S), so integrating the vector field $\vec{F}$ over the surface S is

add up, over the surface, infinitesimal contributions of the form $\vec{F} \cdot d\vec{A}$

We will denote this type of integral as

$$\iint_S \vec{F} \cdot d\vec{A}$$

and refer to this as a *surface integral* for the vector field $\vec{F}$ over the surface S. Another common notation is to express $d\vec{A}$ as $d\vec{A} = \hat{n}\, dA$ where $\hat{n}$ is a unit vector perpendicular to the surface at each point. With this, a surface integral is denoted

$$\iint_S \vec{F} \cdot \hat{n}\, dA.$$

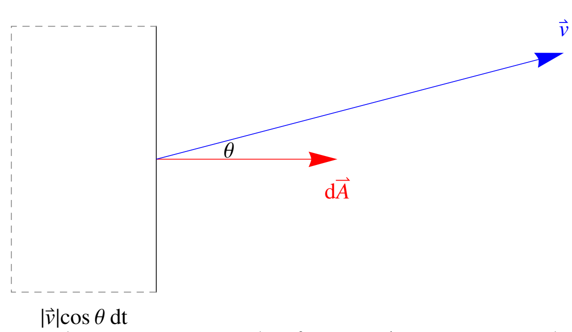
Interpretation

One way to interpret a surface integral is to think of the vector field as representing velocity vectors for a flowing fluid. To emphasize this, let's denote the vector field $\vec{v}$ so that we are considering the integral

$$\iint_S \vec{v} \cdot d\vec{A}$$

Think of the surface S as a rigid mesh held in place with fluid flowing through it. We then have the following interpretation: *The surface integral gives the time rate at which fluid volume flows across this net in the direction of the $d\vec{A}$ vectors.* One way to make this interpretation plausible is to look at units. Since $\vec{v}$ is a velocity, it has units of meters per second (m/s). The infinitesimal area vector $d\vec{A}$ has units of square meters (m^2). So, the product $\vec{v} \cdot d\vec{A}$ has units of cubic meters per second (m^3/s). This is consistent with the claim that each $\vec{v} \cdot d\vec{A}$ represents a contribution to the rate (in time) at which fluid flows across the surface. The integral is a sum of these contributions.

Looking at units makes this interpretation plausible. Using some geometry, we can make a more complete argument. The figure below shows an infinitesimal surface piece viewed from the side along with the corresponding infinitesimal area vector $d\vec{A}$ and the velocity vector $\vec{v}$ for that piece. If θ is the angle between $d\vec{A}$ and $\vec{v}$, then the component of the velocity in the direction perpendicular to the piece is $\|\vec{v}\| \cos\theta$.



Consider an infinitesimal time interval of size dt. During this time, a volume of fluid dV will flow through the surface. All of the fluid that is within a distance

$\|\vec{v}\| \cos\theta \, dt$ from the surface piece when the time interval starts will flow through the surface piece during the time interval. So, the volume dV is equal to that distance multiplied by the area dA which we express as

$$dV = \big(\|\vec{v}\| \cos\theta \, dt\big) \, dA.$$

The *rate* at which fluid volume passes through the surface is dV/dt so we have

$$\frac{dV}{dt} = \frac{\big(\|\vec{v}\| \cos\theta \, dt\big) \, dA}{dt} = \big(\|\vec{v}\| \cos\theta\big) \, dA = \vec{v} \cdot d\vec{A}.$$

With this, we see that we can interpret $\vec{v} \cdot d\vec{A}$ as the rate at which fluid volume is flowing through an infinitesimal piece of the surface. Adding up contributions of this form over the whole surface gives the total rate at which fluid volume flow through the surface.

In the sciences, a rate at which some quantity passes through a surface is called a *flux* for that quantity. (In the interpretation above, that quantity is fluid volume.) For this reason, a surface integral is sometimes called a *flux integral*.

Computing surface integrals

In computing line integrals, the general plan is to express everything in terms of two variables. This is a reasonable thing to do because a surface is a two-dimensional object. The essential things are to determine the form of $d\vec{A}$ for the surface S and the vectors $\vec{F}$ along the surface S, all in terms of two variables. How to proceed depends on how we describe the surface. The solution to the following example illustrates one approach

Example

Compute the line integral of $\vec{F}(x,y,z) = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ for the surface S that is the piece of the plane $12x - 6y + 3z = 24$ with $x \geq 0$, $y \leq 0$, and $z \geq 0$ oriented so that area vectors have a positive $\hat{k}$ component.

To get started, you should draw a picture showing the surface and a few of the vector field outputs along the surface.

From the equation of the plane, we compute

$$12\,dx - 6\,dy + 3\,dz = 0.$$

This relates small displacements dx, dy, and dz along the plane.

To generate one family of curves on the surface, we can use $y = $ constant. This gives us $dy = 0$. Using this in the above relation and solving for dz gives $dz = -4dx$. We can now use these to get

$$d\vec{r}_1 = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k} = dx\,\hat{\imath} + 0\,\hat{\jmath} - 4dx\,\hat{k} = \big(\hat{\imath} - 4\,\hat{k}\big)\,dx.$$

To generate the other family of curves on the surface, we can use $x = $ constant. This gives us $dx = 0$. Using this in the previous relation and solving for dz gives $dz = 2dy$. We can now use these to get

$$d\vec{r}_2 = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k} = 0\,\hat{\imath} + dy\,\hat{\jmath} + 2dy\,\hat{k} = \big(\hat{\jmath} + 2\,\hat{k}\big)\,dy.$$

With $d\vec{r}_1$ and $d\vec{r}_2$ in hand, we can compute $d\vec{A}$ as

$$d\vec{A} = d\vec{r}_1 \times d\vec{r}_2 = \left(\hat{\imath} - 4\,\hat{k}\right) \times \left(\hat{\jmath} + 2\,\hat{k}\right) dxdy = \left(4\,\hat{\imath} - 2\,\hat{\jmath} + \hat{k}\right) dxdy.$$

Note that $d\vec{A}$ has a positive $\hat{k}$ component as desired.

We now want to express the vector field outputs along the surface S in terms of the same two variables (x and y in this case) that we have used for $d\vec{A}$. We will use the equation of the plane to solve for z giving

$$z = 8 - 4x + 2y.$$

Thus, on the surface, the vector field has outputs

$$\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + (8 - 4x + 2y)\,\hat{k}.$$

We now compute

$$\begin{aligned}\vec{F} \cdot d\vec{A} &= \left(x\,\hat{\imath} + y\,\hat{\jmath} + (8 - 4x + 2y)\,\hat{k}\right) \cdot \left(4\,\hat{\imath} - 2\,\hat{\jmath} + \hat{k}\right) dxdy \\ &= \left(4x - 2y + (8 - 4x + 2y)\right) dxdy \\ &= 8\,dxdy\end{aligned}$$

The last things we need in order to carry out the integration are the relevant bounds on the variables x and y. The projection of the surface into the xy-plane is a triangular region in the second quadrant of the xy-plane. We can use

$$0 \le x \le 2 \qquad \text{and} \qquad 2x - 4 \le y \le 0$$

to describe this region.

Putting all of this together, we have

$$\iint\limits_S \vec{F} \cdot d\vec{A} = \int_0^2 \int_{4-2x}^0 8\,dxdy = 8(\text{area of triangle in } xy\text{-plane}) = 8(4) = 32.$$

We can (crudely) interpret this as telling us that fluid flow with velocity given by $x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ carries fluid through the given surface at the rate of $32\,\mathrm{m}^3/\mathrm{s}$. [Note: This is not quite right in terms of units because the vector field $x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ does not have the correct units for a velocity vector field.]

Problems: Integrating a vector field over a surface

1. Compute $\iint\limits_S \vec{F} \cdot d\vec{A}$ where $\vec{F} = \hat{\imath} + 2\,\hat{\jmath} + 3\,\hat{k}$ and S is the piece of the plane $x + z = 1$ with $0 \leq x \leq 1$ and $0 \leq y \leq 2$ oriented so that infinitesimal area vectors have a positive $\hat{k}$ component.

   Answer: 8

2. Compute $\iint\limits_S \vec{F} \cdot d\vec{A}$ where $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ and S is the piece of the plane $x + z = 1$ with $0 \leq x \leq 1$ and $0 \leq y \leq 2$ oriented so that infinitesimal area vectors have a positive $\hat{k}$ component.

   Answer: 2

3. Compute $\iint\limits_S \vec{F} \cdot d\vec{A}$ where $\vec{F} = y\,\hat{\imath} - x\,\hat{\jmath} + 0\,\hat{k}$ and S is the piece of the plane $x + z = 1$ with $0 \leq x \leq 1$ and $0 \leq y \leq 2$ oriented so that infinitesimal area vectors have a positive $\hat{k}$ component.

   Answer: 2

4. Compute $\iint\limits_S \vec{F} \cdot d\vec{A}$ where $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + 0\,\hat{k}$ and S is the open right circular cylinder of radius 2 and height 6 centered at the origin with axis along the z-axis oriented so that infinitesimal area vectors point outward (i.e., away from the z-axis).

5. Compute $\iint\limits_S \vec{F} \cdot d\vec{A}$ where $\vec{F} = y\,\hat{\imath} - x\,\hat{\jmath} + 0\,\hat{k}$ and S is the open right circular cylinder of radius 2 and height 6 centered at the origin with axis along the z-axis oriented so that infinitesimal area vectors point outward (i.e., away from the z-axis).

   Answer: 0

6. Compute $\iint\limits_S \vec{F} \cdot d\vec{A}$ where $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ and S is the open right circular cylinder of radius 2 and height 6 centered at the origin with axis along the z-axis oriented so that infinitesimal area vectors point outward (i.e., away from the z-axis).

   Answer: 48π

7. Compute $\iint\limits_S \vec{F} \cdot d\vec{A}$ where $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ and S is the paraboloid $z = x^2 + y^2$ for $0 \leq z \leq 1$ oriented so that infinitesimal area vectors point outward (i.e., away from the z-axis).