More on Equations of Planes

Earlier we saw three "standard" forms for equations of planes

$Ax + By + Cz + D = 0$	standard form
$z = m_x x + m_y y + b$	slopes-intercept form
$z - z_0 = m_x (x - x_0) + m_y (y - y_0)$	point-slopes form

Now that we have the notation for vectors, we can add a new form that does not rely on coordinates. A plane can be specified by giving two facts about the plane: a vector $\vec{n}$ perpendicular to the plane (called a **normal vector** to the plane) and a point P_0 on the plane. The dot product gives us a geometric test for determining if a variable point P is on the plane. Here is the logic of the test:

- P is on the plane if and only if the vector $\overrightarrow{P_0P}$ is parallel to the plane.
- The vector $\overrightarrow{P_0P}$ is parallel to the plane if and only if $\overrightarrow{P_0P}$ is perpendicular to the normal vector $\vec{n}$.
- The vectors $\overrightarrow{P_0P}$ and $\vec{n}$ are perpendicular if and only if their dot product is zero
$$\vec{n} \cdot \overrightarrow{P_0P} = 0.$$

Note that we now have a new form for the equation of a plane: $\vec{n} \cdot \overrightarrow{P_0P} = 0$. This is called the **point-normal** form for the equation of a plane. If we introduce coordinates of points and components of vectors we can see how the point-normal form is tied to the other forms for planar equations. Let P_0 have coordinates (x_0, y_0, z_0), the variable point P have coordinates (x, y, z) and the normal vector $\vec{n}$ have components $\langle n_x, n_y, n_z \rangle$. Thus the vector $\overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle$ and we can rewrite the point-normal form as follows:

$$\begin{aligned} 0 &= \vec{n} \cdot \overrightarrow{P_0P} \\ &= \langle n_x, n_y, n_z \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle \\ &= n_x (x - x_0) + n_y (y - y_0) + n_z (z - z_0) \\ &= n_x x + n_y y + n_z z + (n_x x_0 + n_y y_0 + n_z z_0) . \end{aligned}$$

Note that the last expression is in the form $Ax + By + Cz + D = 0$ and the second to the last expression is two algebra steps away from the slopes-intercept form (if $n_z \neq 0$).

EXAMPLE:
Find the standard form for the equation of the plane that contains the point $(3, -2, 4)$ and has normal vector $\langle 5, 6, 7 \rangle$.

Solution:
Using (x, y, z) as the coordinates of a variable point P we have

$$\begin{aligned} 0 &= \vec{n} \cdot \overrightarrow{P_0P} \\ &= \langle 5, 6, 7 \rangle \cdot \langle x - 3, y + 2, z - 4 \rangle \\ &= 5 (x - 3) + 6 (y + 2) + 7 (z - 4) \\ &= 5x + 6y + 7z - 15 + 12 - 28 \\ & 5x + 6y + 7z - 31 \end{aligned}$$

So the standard form for the equation of this plane is $5x + 6y + 7z - 31 = 0$

Problems on Equations of Planes

1. Use the point-normal equation to determine which, if any, of the of the following points are on the plane that has normal vector $2\widehat{\imath} - \widehat{\jmath} + 6\widehat{k}$ and contains the point $(3, 4, 2)$.

   (a) $(5, -4, 0)$

   (b) $(1, 6, 2)$

   (c) $(2, 8, 3)$

2. Find the slopes-intercept form of the equation that contains the point $(4, 2, -7)$ and has normal vector $5\widehat{\imath} - 3\widehat{\jmath} + 2\widehat{k}$.

3. Find the slopes-intercept form of the equation that contains the point $(4, 2, -7)$ and has normal vector $\vec{n} = \langle -6, 1, 5 \rangle$.

4. Find the standard form of the equation of the plane that contains the point $(6, 3, 0)$ and is parallel to a second plane that has equation $5x + 2y - 9z = 14$.

5. Find the standard form of the equation for the plane that contains the point $(7, -2, 1)$ and is perpendicular to the vector from the origin to that same point.

6. Determine a normal vector for each of the planes in Exercises 2-6 of the first handout on planes.

Solutions

1. $(5, -4, 0), (2, 8, 3)$ on the plane, $(1, 6, 2)$ not

2. $z = -\frac{5}{2}x + \frac{3}{2}y$

3. $z = \frac{6}{5}x - \frac{1}{5}y - 7$

4. -

5. $7x - 2y + z - 54 = 0$