Divergence of a vector field

Flux

Given a vector field $\vec{F}$ and an oriented surface S in space, we can think of the surface integral $\iint_S \vec{F} \cdot d\vec{A}$ as a *flux*. In this interpretation, we think of $\vec{F}$ as the velocity field of a fluid flow and think about the surface S as a rigid net in this flow. At each point on the surface, the components of $\vec{F}$ tangent to the surface do not contribute to moving fluid across the net. Only the component of $\vec{F}$ that is perpendicular to the surface is relevant. The area element vector $d\vec{A}$ points perpendicular to the surface at each point so the dot product $\vec{F} \cdot d\vec{A}$ gives us the component of $\vec{F}$ perpendicular to the surface multiplied by the (infinitesimal) area dA. This is the rate at which fluid volume is flowing through the surface at each point. The surface integral $\iint_S \vec{F} \cdot d\vec{A}$ is a summing up of these contributions and so gives us the total rate at which fluid volume flows across the surface.

Divergence as flux density

Start with a vector field $\vec{F}$ and focus on a point $\mathcal{P}$ in the domain of the vector field. Imagine a small solid region that contains $\mathcal{P}$. (You can think of a rectangular box or a sphere if it helps to be specific about the shape.) We will use ΔD to denote this solid region. Here Δ doesn't mean "a small change in" but serves to remind us that the region is small. Let ΔV be the volume of ΔD. Let ΔS be the closed surface that is the boundary of this solid region. Orient the area element vectors $d\vec{A}$ for ΔS to be pointing outward.

Both the flux $\oiint_{\Delta S} \vec{F} \cdot d\vec{A}$ and the volume ΔV will go to zero as we shrink the solid region ΔD down to the point $\mathcal{P}$. However, the limit of the ratio

$$\frac{\oiint_{\Delta S} \vec{F} \cdot d\vec{A}}{\Delta V}$$

might exist. If so, this limit is the *volume flux density*. That is, the limit of the ratio is the flux per unit volume. We define the *divergence of the vector field* $\vec{F}$ in terms of this flux density.

The divergence of $\vec{F}$ at $\mathcal{P}$ is defined as the flux density at a point $\mathcal{P}$. That is,

$$\operatorname{div} \vec{F}(\mathcal{P}) = \lim_{\text{“}\Delta D \to \mathcal{P}\text{''}} \frac{\oiint_{\Delta S} \vec{F} \cdot d\vec{A}}{\Delta V}$$

Example 1

Compute the divergence of the vector field $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ *at the origin* $(0,0,0)$.

This vector points radially out from the origin so a convenient choice of a solid region ΔD is a ball of radius R centered at the origin. The boundary ΔS of this solid region is the sphere of radius R and the volume of the region is $\Delta V = 4\pi R^3/3$. We have previously computed the flux for this vector field through the sphere of radius R and found

$$\oiint_{\text{sphere}} \vec{F} \cdot d\vec{A} = 4\pi R^3.$$

So, we can form the ratio

$$\frac{\oiint_{\Delta S} \vec{F} \cdot d\vec{A}}{\Delta V} = \frac{4\pi R^3}{4\pi R^3/3} = 3$$

We thus have

$$\operatorname{div} \vec{F}(0,0,0) = \lim_{\text{“}\Delta D \to \mathcal{P}\text{''}} \frac{\oiint_{\Delta S} \vec{F} \cdot d\vec{A}}{\Delta V} = \lim_{R \to 0} 3 = 3.$$

So, the flux density for $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ at the origin $(0,0,0)$ is positive. In a fluid flow interpretation, we can think of this as saying the fluid is being injected into the flow at $(0,0,0)$.

Computing divergence directly as a flux density is only feasible in cases with lots of symmetry. We now turn attention to a more efficient way to compute the divergence of a vector field.

An expression for divergence in cartesian coordinates

From the definition of divergence in terms of flux density, we learn what divergence tells us about the vector field. However, computing the divergence of a vector from this definition is difficult. We'll next look at getting an expression for the divergence in terms of partial derivatives with respect to cartesian coordinates.

Let the vector field $\vec{F}$ be given in cartesian coordinates by

$$\vec{F}(x,y,z) = P(x,y,z)\,\hat{\imath} + Q(x,y,z)\,\hat{\jmath} + R(x,y,z)\,\hat{k}.$$

We will compute the divergence using a rectangular box with one corner at the point $\mathcal{P}(x,y,z)$ and the edges of the box parallel to the coordinate axes as shown in Figure 1. Let Δx, Δy, and Δz be the edge lengths parallel to each of the coordinate axes. This rectangular box has six sides. Let $\Delta\vec{A}_1, \Delta\vec{A}_2, \ldots, \Delta\vec{A}_6$ be the area vectors for these six sides with the first two for the faces parallel to the yz-plane, the next two parallel to the xz-plane, and the last two parallel to the xy-plane. We then have, for example, $\Delta\vec{A}_1 = -\Delta y \Delta z\,\hat{\imath}$.

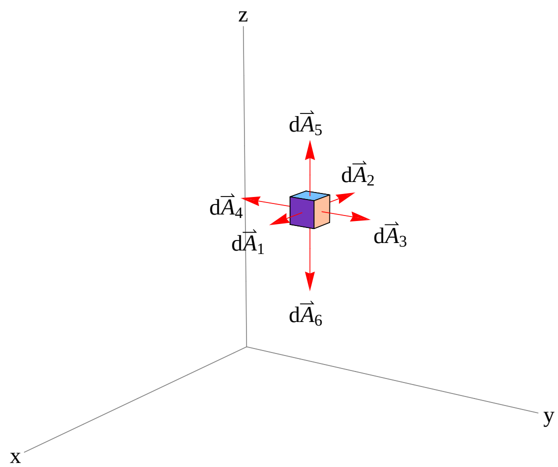

Figure 1. The rectangular box and normal vectors used in deriving a component expression for divergence.

For this geometry, the surface integral over the boundary of the box is approximated by a sum of six terms:

$$\oiint_{\Delta S} \vec{F} \cdot d\vec{A} \approx \vec{F}_1 \cdot \Delta\vec{A}_1 + \vec{F}_2 \cdot \Delta\vec{A}_2 + \vec{F}_3 \cdot \Delta\vec{A}_3 + \vec{F}_4 \cdot \Delta\vec{A}_4 + \vec{F}_5 \cdot \Delta\vec{A}_5 + \vec{F}_6 \cdot \Delta\vec{A}_6$$

with one term for each face of the box. In each term, the vector field is evaluated at a point on the corresponding face. For example, on the first face, we have

$$\vec{F}_1 \cdot \Delta\vec{A}_1 = -\vec{F}(x,y,z) \cdot \Delta y \Delta z\, \hat{\imath} = -P(x,y,z)\, \Delta y \Delta z.$$

For the opposite face, we have $\Delta\vec{A}_2 = +\Delta y \Delta z\, \hat{\imath}$, and

$$\vec{F}_2 \cdot \Delta\vec{A}_2 = \vec{F}(x+\Delta x,y,z) \cdot \Delta y \Delta z\, \hat{\imath} = P(x+\Delta x,y,z)\, \Delta y \Delta z.$$

Pairing the opposite sides parallel to the xz-plane and xy-plane in a similar fashion, we have

$$\oiint_{\Delta S} \vec{F} \cdot d\vec{A} \approx [P(x+\Delta x,y,z) - P(x,y,z)]\, \Delta y \Delta z + [Q(x,y+\Delta y,z) - Q(x,y,z)]\, \Delta x \Delta z \\ + [R(x,y,z+\Delta z) - R(x,y,z)]\, \Delta x \Delta y.$$

To get the flux density at $\mathcal{P}(x,y,z)$, we divide by the volume $\Delta V = \Delta x \Delta y \Delta z$ and take a limit as $\Delta D \to \mathcal{P}$. This is equivalent to $\Delta x, \Delta y, \Delta z \to 0$. The flux density is thus

$$\lim_{``\Delta D \to \mathcal{P}''} \frac{\oiint_{\Delta S} \vec{F} \cdot d\vec{A}}{\Delta V}$$

$$= \lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{[P(x+\Delta x,y,z)-P(x,y,z)]\Delta y \Delta z + [Q(x,y+\Delta y,z)-Q(x,y,z)]\Delta x \Delta z + [R(x,y,z+\Delta z)-R(x,y,z)]\Delta x \Delta y}{\Delta x \Delta y \Delta z}$$

$$= \lim_{\Delta x, \Delta y, \Delta z \to 0} \left[\frac{P(x+\Delta x,y,z)-P(x,y,z)}{\Delta x} + \frac{Q(x,y+\Delta y,z)-Q(x,y,z)}{\Delta y} + \frac{R(x,y,z+\Delta z)-R(x,y,z)}{\Delta z} \right]$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

With this, we have the following result.

In cartesian coordinates, the divergence of

$$\vec{F} = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}$$

is given by

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Example 2

Compute the divergence of the vector field $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$.

Using the coordinate expression for divergence, we have

$$\operatorname{div} \vec{F}(x,y,z) = \frac{\partial}{\partial x}[x] + \frac{\partial}{\partial z}[z] + \frac{\partial}{\partial z}[z] = 1 + 1 + 1 = 3.$$

So, the divergence of $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ is equal to 3 at all points (x,y,z). Note that this is consistent with what we got in Example 1 for the divergence of the same vector field at $(0,0,0)$. In a fluid flow interpretation, we can think of this as saying that fluid is being added to the flow at a certain rate at every point in space.

The operator point of view

When we use the derivative operator d/dx, we think of it as meaning "take the derivative with respect to x of whatever follows". So, for example, we read

$$\frac{d}{dx}[\sin x]$$

as "take the derivative with respect to x of $\sin x$". We record the result of doing this as

$$\frac{d}{dx}[\sin x] = \cos x.$$

In similar fashion, the operator $\partial/\partial x$ means "take the partial derivative with respect to x of whatever follows".

We now introduce the *vector operator* $\vec{\nabla}$. In cartesian coordinates, this operator is

$$\vec{\nabla} = \hat{\imath}\,\frac{\partial}{\partial x} + \hat{\jmath}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}.$$

If we apply this operator to a function f of three variables, we get

$$\vec{\nabla} f = \left(\hat{\imath}\,\frac{\partial}{\partial x} + \hat{\jmath}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)[f] = \hat{\imath}\,\frac{\partial f}{\partial x} + \hat{\jmath}\,\frac{\partial f}{\partial y} + \hat{k}\,\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\,\hat{\imath} + \frac{\partial f}{\partial y}\,\hat{\jmath} + \frac{\partial f}{\partial z}\,\hat{k}$$

This is just our familiar notation for the *gradient of* f.

Now consider $\vec{\nabla}$ acting on a vector field $\vec{F} = P\,\hat{\imath} + Q\,\hat{\jmath} + R\,\hat{k}$. Since both $\vec{\nabla}$ and $\vec{F}$ are vector objects, one way to let $\vec{\nabla}$ act on $\vec{F}$ is to *dot* the two. This gives us

$$\vec{\nabla} \cdot \vec{F} = \left(\hat{\imath}\,\frac{\partial}{\partial x} + \hat{\jmath}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right) \cdot \left(P\,\hat{\imath} + Q\,\hat{\jmath} + R\,\hat{k}\right) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Notice that this is precisely the *divergence of* $\vec{F}$. So, we can write

$$\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F}.$$

Example 3

Compute the divergence of the vector field $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ using the operator style.

Note that we are just repeating Example 2 using a different style. We have

$$\begin{aligned}\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F} &= \left(\hat{\imath}\,\frac{\partial}{\partial x} + \hat{\jmath}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right) \cdot \left(x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}\right) \\ &= \frac{\partial}{\partial x}[x] + \frac{\partial}{\partial z}[z] + \frac{\partial}{\partial z}[z] = 1 + 1 + 1 = 3.\end{aligned}$$

The operator style will seem more useful once we use it in expressing the *curl* of a vector field.

Problems: divergence of a vector field

1. Compute the divergence of the vector field $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + 0\,\hat{k}$ at the origin $(0,0,0)$ directly as a flux density. For this, use a region ΔD in the form of a solid cylinder centered at the origin of radius R and height H.

   (a) Use a geometric argument to compute the flux $\oiint_{\Delta S} \vec{F} \cdot d\vec{A}$ of the vector field through the cylinder surface.

   (b) Write down the volume ΔV of this cylinder.

   (c) Form the ratio $\oiint_{\Delta S} \vec{F} \cdot d\vec{A} / \Delta V$.

   (d) Evaluate the limit of the ratio in (c) as $R \to 0$ and $H \to 0$.

For each of the following vector fields, compute the divergence. Evaluate the divergence at a few points and give an interpretation for each value.

2. $\vec{F} = x\,\hat{\imath} + y\,\hat{\jmath} + 0\,\hat{k}$

3. $\vec{F} = -y\,\hat{\imath} + x\,\hat{\jmath} + 0\,\hat{k}$

4. $\vec{F} = x^2\,\hat{\imath} + y^2\,\hat{\jmath} + z^2\,\hat{k}$

5. $\vec{F} = z\sin(xy)\,\hat{\imath} + (x+y)\,\hat{\jmath} + ze^x\,\hat{k}$

6. $\vec{F} = \dfrac{x\,\hat{\imath} + y\,\hat{\jmath}}{\sqrt{x^2+y^2}}$

7. $\vec{F} = \dfrac{x\,\hat{\imath} + y\,\hat{\jmath}}{x^2+y^2}$

8. $\vec{F} = \dfrac{x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}}{\sqrt{x^2+y^2+z^2}}$

9. $\vec{F} = \dfrac{x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}}{(x^2+y^2+z^2)^{3/2}}$