Applied optimization problems

1. Find three positive numbers so that the first plus twice the second plus three times the third equals 26 and the product of the three is maximized.

   Answer: 26/3, 13/3, and 26/9

2. The United States Postal Service web site states "The maximum size for most mailpieces is 108 inches in combined length and girth." Find the maximum volume of a rectangular package allowed under this condition.
   Note: Length is the longest of the three dimensions. Girth is the perimeter of the rectangle with the two shorter dimensions.

   Answer: 18 inches $\times$ 18 inches $\times$ 36 inches

3. In your job at the container factory, you are asked to design a rectangular box with volume 500 cm^3. The material for the sides and bottom costs \$0.05 per cm^2 while the material for the top costs \$0.15 per cm^2. What dimensions do you recommend to minimize the total material cost?

   Answer: base: $5\sqrt[3]{2}$ cm $\times$ $5\sqrt[3]{2}$ cm, height: $10\sqrt[3]{2}$ cm

4. Still in your job at the container factory, you are asked to address this type of problem more generally. The design specifications call for a rectangular box of total volume V (in cm^3). The material for the top costs a (in dollars per cm^2), the material for the bottom costs b (in dollars per cm^2), and the material for the sides costs c (in dollars per cm^2). What dimensions do you recommend to minimize the total material cost?
   Note: You should expect the minimizing dimensions to be in terms of the parameters in the problem (V, a, b, and c).

5. You own a manufacturing company that produces two versions of a tablet computer, a standard version and an advanced version. Basic economic theory tells us that the price of each will decrease as the available quantities increase. Let q_1 and q_2 be the available quantities of the standard and advanced versions, respectively. Let p_1 and p_2 be their selling prices in dollars. Your marketing department has produced the following model for the relationships among these variables:

   $$p_1 = 400 - 0.1q_1 - 0.04q_2 \qquad \text{and} \qquad p_2 = 500 - 0.02q_1 - 0.6q_2$$

   The cost for you to manufacture and distribute each unit of the standard version is \$50 while the cost for the advanced unit is \$65. How many units of each should you make available in order the maximize your profit?
   Note: Don't worry if your results are not whole numbers.

   Answer: about 1666 standard units and 279 advanced units

6. Consider studying consumer utility for a bundle of goods consisting of pizza, coffee, and textbooks. Let p, c, and t be the amounts of pizza, coffee, and textbooks in the bundle. Assume the utility for this bundle is given by

$$U(p, c, t) = p^{1/3} c^{1/2} t^{1/6}.$$

Suppose the unit costs of pizza, coffee, and textbooks are \$10, \$2, and \$80, respectively. A consumer has a total of \$1000 available. Find the combination of pizza, coffee, and textbooks that maximizes the utility for this consumer. Note: Don't worry if your results are not whole numbers.

Answer: 100/3 pizzas, 250 coffees, and 25/12 textbooks