Sets and Logic — Handout 01

Set Basics

- Notation for standard sets of numbers: $\mathbf{C}, \mathbf{R}, \mathbf{Q}, \mathbf{Z}, \mathbf{N}$
- Standard operators on sets: $\in, \cup, \cap, \subseteq, \notin$
- "Set Builder" notation: {Universal set | defining restriction}
    1. **Example:** The set of even integers: $\{n \in \mathbf{Z} \mid n = 2k \text{ and } k \in \mathbf{Z}\}$
    2. **Example:** The set of Real-valued functions whose domain is the set of Real numbers and whose graph passes through the point $(2,5)$: $\{f : \mathbf{R} \longrightarrow \mathbf{R} \mid f(2) = 5\}$.

Logical Operators and their Truth Tables

1. **Not:** (negation): ˜
2. **And** (Conjunction): $\wedge$
3. **Or:** (Disjunction): $\vee$
4. **Conditional** (Implication): $\Longrightarrow$
5. **Equivalence** (If and only if): $\Longleftrightarrow \quad (\equiv)$

Tautologies / Contradictions

1. $p \wedge \sim p$
2. $p \vee \sim p$
3. $\sim\sim p \Longleftrightarrow p$
4. $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$
5. $(p \vee q) \Longleftrightarrow (\sim p) \wedge (\sim q)$
6. $(p \wedge q) \iff (\tilde{}p) \vee (\tilde{}q)$
7. $(p \Longrightarrow q) \Longleftrightarrow (\sim q) \Longrightarrow (\sim p)$ contrapositive
8. $(p \Longrightarrow q) \Longleftrightarrow (\sim p) \vee q$
9. $((P \wedge \tilde{}Q) \Rightarrow (R \wedge \tilde{}R)) \Longleftrightarrow (P \Rightarrow Q)$
10. $((p \Longrightarrow q) \Longrightarrow (r \Longrightarrow s)) \iff ((p \Longrightarrow q) \wedge r) \Longrightarrow s$

Quantifiers

Universal: $\forall$ **Example:** $\forall x \in \mathbf{R} \quad x^2 + 1 > 0$ is a true statement

Existential: $\exists$ **Example:** There is an integer solution to $x^2 + 5x + 6 = 0$ is a true statement. $(x = -2)$

Negation of quantifiers $\tilde{}\exists x\ (p(x))$ means $\forall x\ \tilde{}p(x)$

Proof Methods

Direct Proof of $H \Longrightarrow C$ or $H \Longrightarrow C_1 \wedge C_2$

1. Start with the (conjoined) hypotheses of H
2. Use nothing but logical steps See below.
3. Deduce C. (Deduce each of the C_i)

Use of the Contrapositive to prove $H \implies C$ Uses the tautology $(H \implies C) \iff (\sim C) \implies (\sim H)$

1. Start with (conjoined) statements of $\sim C$
2. Use nothing but logical steps
3. Deduce $\sim H$

Proof by Contradiction of $H \implies C$ Uses Tautology $(((H \wedge (\sim C))) \implies (D \wedge (\sim D))) \implies C$

1. Start with $(\sim C)$
2. Use H and nothing but logical steps to get $(D \wedge (\sim D))$
3. Deduce $(\sim\sim C)$

How to deal with conjunctions and disjunctions

Disjoined Hypotheses $H_1 \vee H_2 \implies C$ **Uses the Tautology ...**

1. Do it by cases: Prove the 2 individual implications $H_i \implies C$

Disjoined Conclusions $H \implies C_1 \vee C_2$ **Uses the Tautology** $[H \implies (C_1 \vee C_2)] \iff [(H \wedge \sim C_1) \implies C_2]$

1. Start with C and the negation of all but one C_i
2. Deduce the last C.

How to prove Universal statements $\forall x \ (p(x) \implies q(x))$

1. Start with an **arbitrary** element x in the universal set X
2. Show that, using only the properties of being in X $p(x) \implies q(x)$
3. **Example:** If $x > 1$ then $x^2 > x$.
   Proof: Let x be an arbitrary number bigger than 1.

How to prove Existential Statements $\exists x \ p(x)$

1. Best approach is to **actually exhibit** an instance of x.
2. Or do a proof by contradiction.

Forward-Backward method for doing proofs

Basic (Named) Rules of Inference

1. Modus Ponens (mode that affirms)(mode that affirms by affirming) $((p \implies q) \wedge p) \implies q$
2. Syllogism $((p \implies q) \wedge (q \implies r)) \implies (p \implies r)$
3. Contrapositive $(p \implies q) \iff ((\sim q) \implies (\sim p))$
   (a) converse, obverse
4. Modus Tollens (mode that denies)("the way that denies by denying") $((p \implies q) \wedge (\sim q)) \implies (\sim p)$
5. Contradiction $(((p \wedge (\sim q))) \implies (r \wedge (\sim r))) \implies q$