Mathematics 433   Fall 2010   Problems to Turn in: 10

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Name

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use the results of discussion, a text, notes, or technology. **Only write on one side of each page.**

"In 1910 the mathematician Oswald Veblen and the physicist James Jeans were discussing the reform of the mathematical curriculum at Princeton University. 'We may as well cut out group theory,' said Jeans. 'That is a subject which will never be of any use to physics.' It is not recorded whether Veblen disputed Jeans' point, or whether he argued for the retention of group theory on purely mathematical grounds. All we know is that group theory continued to be taught. And Veblen's disregard for Jeans' advice continued to be of some importance to the history of science at Princeton. By the irony of fate group theory later grew into one of the central themes of physics, and it still dominates the thinking of all of us who are struggling to understand the fundamental particles of nature." Freeman J. Dyson SCIENTIFIC AMERICAN, Sep, 1964

Problems

1. Show that a homomorphism with domain a cyclic group is compeletely determined by its action on a generator of the domain.

2. Prove or disprove: If H is a normal subgroup of G such that both H and G/H are abelian, then G is abelian.

3. Prove or disprove: If both H and G/H are cyclic, then G is cyclic.

4. If G is a group with exactly one subgroup H of order k, prove that H is normal in G.

5. Prove or disprove: $\mathbf{Q}/\mathbf{Z} \cong \mathbf{Q}$.

6. Let G be a finite group and N a normal subgroup of G. Let $\phi : G/N$ be the homorphism given by $\phi(g) = gN$ and let H' be a subroup of G/N. Prove that the preimage of H', $\phi^{-1}(H')$ is a subgroup of G with order $|H| \cdot |N|$.

7. If H and K are normal subgroups of G and $H \cap K = \{e\}$, prove that G is isomorphic to $G/H \times G/K$.

8. Let $\phi : G_1 \to G_2$ be a surjective group homomorphism. Let H_1 be a normal subgroup of G_1 and suppose that $\phi(H_1) = H_2$. Prove or disprove that $G_1/H_1 \cong G_2/H_2$.

9. Let G be a group and $i_g : G \to G$ defined by $i_g(x) = gxg^{-1}$ an inner automorphism of G, and define a map
$$G \to Aut(G)$$
by
$$g \mapsto i_g$$
Prove that this map is a homomorphism with image $Inn(G)$ and kernel $Z(G)$. Use this result to conclude that $G/Z(G) \cong Inn(G)$

10. Let G be a group and let $G' = \langle aba^{-1}b^{-1} \rangle$ designate the set of all finite products of elements of the group G that have the form $aba^{-1}b^{-1}$. The subgroup G' of G is called the commutator subgroup of G.

(a) Prove that G' is a subgroup of G.

(b) Prove that G' is normal in G

(c) Let N be a normal subgroup of G. Prove that G/N is abelian if and only if N contains the commutator subgroup G'.