Mathematics 433 Fall 2010 Problems to Turn in: 9

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Name

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use the results of discussion, a text, notes, or technology. **Only write on one side of each page.**

"*A life spent making mistakes is not only more honorable, but more useful than a life spent doing nothing.*"
– George Bernard Shaw

Problems

1. Do both of the following:
   (a) Prove that $O(2)$ is not a normal subgroup of M where M denotes the group of all rigid motions of the plane.
   (b) Let $SO(2)$ denote the subset of orientation-preserving motions of the plane. Prove $SO(2)$ is a normal subgroup of M and determine its index in M.

2. For those of you who know a bit of complex variables.
   (a) Write the formulas for the motions t_a, ρ_θ and r in terms of the complex variables $z = x + iy$.
   (b) Show every motion has the form $m(z) = \alpha z + \beta$ or $m(z) = \alpha\overline{z} + \beta$, where α, β are complex numbers with $|\alpha| = 1$.
   (c) Find an isomorphism from the group SO to the subgroup of $GL(2, \mathbf{C})$ of matrices of the form $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ with $|a| = 1$.

3. With each of the patterns shown on the sheet of figures labelled "Problem 8.3", find a pattern with the same type of symmetry as those on the accompanying handout (the page numbered 173).

4. Given the subgroup $H = \{1, x^5\}$ of the dihedral group $D_{10.}$
   (a) Explicitly compute the cosets of H in D_{10}.
   (b) Prove that D_{10}/H is isomorphic to D_5.
   (c) Is D_{10} isomorphic to $D_5 \times H$?

5. List all symmetries of the following figures (found on the recent handout on symmetry).
   (a) Figure 1.4
   (b) Figure 1.5
   (c) Figure 1.6
   (d) Figure 1.7

6. Prove every finite subgroup of M is a conjugate subgroup of one of the standard subgroups listed in the corollary to the Classification of Finite Symmetry Groups Theorem stated below.

(a) **Corollary 1** *Let G be a finite subgroup of the group of motions M . If coordinates are introducted suitably, then G becomes one of the groups C_n or D_n, where C_n is generated by ρ_θ, $\theta = 2\pi/n$ and D_n is generated by ρ_θ and r.*

7. Find all proper normal subgroups N and identify the corresponding quotient groups D_k/N of the groups D_{13} and D_{15}.

8. Let G be a subgroup of M that contains rotations about two different points. Prove algebraically that G contains a translation.

9. Prove the group of symmetries of the frieze pattern

$$\cdots \text{EEEEEEEE} \cdots$$

is isomorphic to the direct product $C_2 \times C_\infty$ of a cyclic group of order 2 and an infinite cyclic group.

10. Let G be the group of symmetries of the frieze pattern

$$\cdots \subset \supset \subset \supset \subset \supset \cdots$$

(a) Determine the point group $\bar{G}$ of G.

(b) For each element $\bar{g}$ of $\bar{G}$, and each element g of G which represents $\bar{g}$, describe the action of g geometrically.

(c) Let H be the subgroup of translations in G. Determine $[G : H]$.

11. Let G be a discrete group in which every element is orientation-preserving. Prove the point group $\bar{G}$ is a cyclic group of rotations and there is a point p in the plane such that the set of group elements which fix p is isomorphic to $\bar{G}$.

12. Recall that M is the group of rigid motions of the two-dimensional plane. In this problem you investigate the rigid motions of a one-dimensional line.

Let N denote the group of rigid motions of the line $l = \mathbf{R}^1$.Some elements of N are

$$t_a \text{ where } t_a(x) = x + a \text{ and } s \text{ where } s(x) = -x.$$

(a) Show that $\{t_a, t_a s : a \in \mathbf{R}^1\}$ are all of the elements of N,and describe their actions on l geometrically. [Note that $|N|$ is infinite since there is a distinct t_a for each real number a.]

(b) Compute the products $t_a t_b$, st_a, ss.

(c) Find all discrete subgroups of N which contain a translation. It will be convenient to choose your origin and unit length with reference to the particular subgroup. Prove your list is complete.

13. Prove

(a) If the point group of a lattice group G is $\bar{G} = C_6$, then $L = L_G$ is an equilateral triangular lattice, and G is the group of all rotational symmetries of L about the lattice points.

(b) If the point group of a lattice group G is $\bar{G} = D_6$, then $L = L_G$ is an equilateral triangular lattice, and G is the group of all symmetries of L.

Figure 1:

Figure 1:

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Figure 2: